④ Applications of LU

(1) Solve linear equations. How?

$$PA = LU \quad \rightsquigarrow \quad A = P^T LU$$

$$Ax = b \rightsquigarrow \cancel{P}\,\cancel{P^T} LUx = Pb \;\Big|\; P\cdot$$

$$\underbrace{LUx}_{y} = Pb \quad \to \text{use fw. subst to find } y$$

$$Ux = y \quad \to \text{use bw subst to find.}$$

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(2) Solve a matrix equation. How?

$$AX = B \quad \xrightarrow{\text{example}} \quad AX = I$$

here: X inverse

Assume: A, X, B square

$\boxed{X}$ unkown

$\boxed{A}\,\boxed{B}$

known

columns of B: $b_1, b_2, \ldots$

columns of X: $x_1, x_2, \ldots$

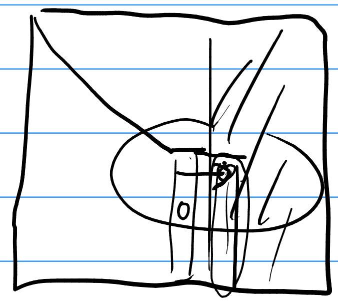
Solve $Ax_1 = b_1$ for x_1

$Ax_2 = b_2$ for x_2

$$PA = LU \leftarrow \text{cost}: O(n^3)$$

$$+\, n \cdot (\underbrace{\text{fw} + \text{bw subst}}_{O(n^2)}) \leftarrow O(n^3)$$

(3) Find row echelon form



$$\overbrace{M_3 \, P_3 \, M_2 \, P_2 \, M_1 \, P_1}^{M} \, A$$

$$L_3 L_2 L_1 \; P_3 P_2 P_1$$

$$L_2 = P_3 M P_3^{-1}$$

So we can get an invertible matrix M so that

$MA = U$ upper echelon form

$PA = LU$ not upper echelon form

$A = M^{-1} U$

(4) Find the basis of a span. How?

Given: $\vec{x}_1, \ldots, \vec{x}_n$ lin *de*pendent

Want: $\vec{y}_1, \ldots, \vec{y}_{n-k}$ lin *in*dependent

$$A = \begin{pmatrix} — & \vec{x}_1 & — \\ & \vdots & \\ — & \vec{x}_n & — \end{pmatrix}$$

Find echelon form: $MA = U$

(5) Find the determinant of a matrix. How?

$PA = LU \longleftarrow \det(AB) = \det(A)\det(B)$

$\det(P) \cdot \det(A) = \det(L) \cdot \det(U)$

± 1 | 1 | multiply up the diagonal

det(triangular matrix)

= product of diagonal entries

"solve" for me

(5) We'd like to find the rank* of a matrix. Is that possible using a computer?

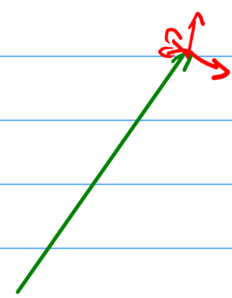
Two randomly drawn vectors almost surely do not point in the same direction.

Computers do not represent numbers exactly.

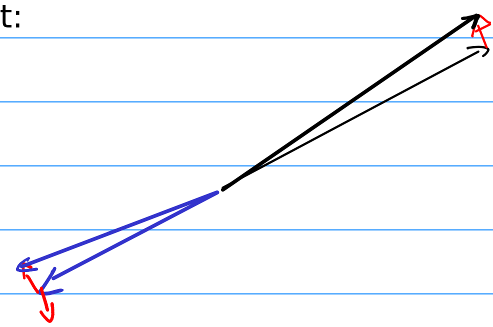
Every FP number: $\underbrace{3.14159}_{\text{good}}\underbrace{7777777}_{\text{junk}}$

Every stored number: True value + small error

random



Suppose we want to test whether two vectors are linearly dependent:



Lesson: we can't hope for exact equality in testing for linear dependence

*rank: Number of linearly independent rows/columns

Suppose we take that into account. How would we compute the rank?

To compute the rank, find the row echelon form

and test the bottom rows for having a norm greater or

or equal to a tolerance---and then only consider rows with

norm greater than the tolerance "non-zero."

Lesson: When finding the rank computationally, you *must*
specify a tolerance.

(6) Finding the nullspace of a matrix A

upper echelon

$$MA^T = U$$

$$\hookrightarrow \quad A^T = M^{-1}U$$

$$\iff A = U^T M^{-T}$$

$U \to$ echelon form  $U^T \to$ 

$$N(U^T) = \text{span}\{(0,0,\dots,0,1,0,\dots,0),$$

$$\to \vec{x} = (\quad 1 \quad), \; (\quad 1 \dots)$$

$$A = U^T(M^{-T} \quad)$$

Looking for $M^{-T}\vec{y} = \vec{x}$

$$\to \quad \vec{y} = M^T\vec{x}$$

$$N(A) = M^T N(U^T)$$

5

Orthogonality and QR

The 'linear algebra way' of talking about "angle" and "similarity" between two vectors is called "inner product". We'll define this next.

So, what is an inner product?

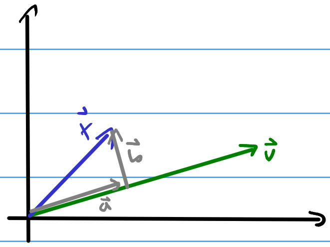
Can you give an example?

Is any old inner product also "linear" in the second argument?

Do inner products relate to norms at all?

Tell me about orthogonality.

What if I've got two vectors that are not orthogonal, but I'd like them to be?



Given: $\vec{x}, \vec{v}$

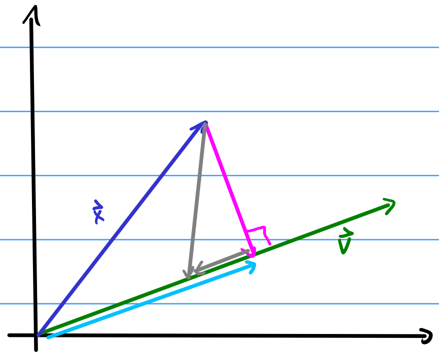
Want: $\vec{a}, \vec{b}$

with $\vec{a} = \alpha \vec{v}$

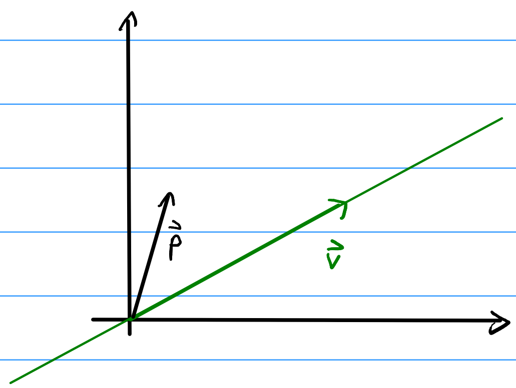
$\vec{b} \perp \vec{v}$

$\vec{x} = \vec{a} + \vec{b}$

In this situation, where is the closest point to $\vec{x}$ on the line $\beta\vec{v}$?



How does the point-normal form (of a line) work?



What's an orthogonal basis?

What's an orthonormal basis (or "ONB")?

For some given vector $\vec{x}$, how do I find coefficients with respect to an ONB?

$$\vec{x} = \underline{\qquad}\,\vec{b}_1 + \underline{\qquad}\,\vec{b}_2 + \underline{\qquad}\,\vec{b}_3 + \cdots + \underline{\qquad}\,\vec{b}_n$$

Can we build a matrix that computes those coefficients for us?

What else is true for a orthogonal matrix Q?

What if Q contains a few zero columns instead of orthonormal vectors?

Orthonormal vectors seem very useful. How can we make more than two?