(6) Finding the nullspace of a matrix A

$$\begin{pmatrix} 1 & 2 & \overset{\downarrow}{0} \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \textcircled{1} \end{pmatrix}$$

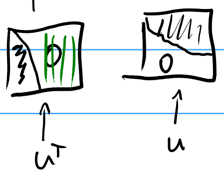
$M A^T = U$ ← U is in upper echelon form

$\Leftrightarrow \quad A^T = M^{-1} U$ $\qquad N(A) = N(U^T M^{-T}) = N(BC$

$\Leftrightarrow \quad A = U^T M^{-T}$ $\qquad A y = 0$

invertible

$U^T \underbrace{M^{-T} y} = 0$



$U^T x = 0$

Can find N(U^T) by vectors x = (0,0,...,0,1,0,...0)

$x = \boxed{M^{-T} y} \quad | \cdot M^T$

$y = M^T x \qquad \Rightarrow \qquad N(A) = M^T N(U^T)$

⑤ Orthogonality and QR

The 'linear algebra way' of talking about "angle" and "similarity" between two vectors is called "inner product". We'll define this next.

So, what is an inner product?

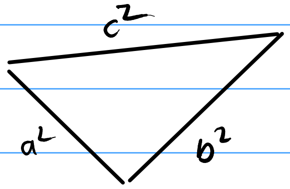
$$f(\vec{x}, \vec{y}) \to \text{number}$$

$$f(x, x) \geq 0$$

$$f(\vec{x}, \vec{y}) = 0 \iff x, y \quad \text{orthogonal}$$

$$\left.\begin{aligned} \vec{x}' &= \vec{x} \\ \vec{y}' &= \vec{y} - \frac{f(\vec{x}, \vec{y})}{f(\vec{x}, \vec{x})}\vec{x} \end{aligned}\right\} \Rightarrow \vec{x}' \perp \vec{y}' \iff f(\vec{x}', \vec{y}') = 0$$

Pythagorean theorem: $\vec{x} \perp \vec{y}$: $\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$



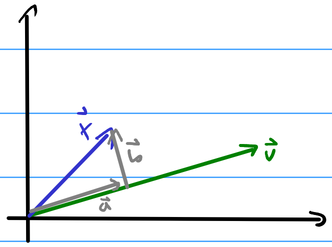
Can you give an example?

Is any old inner product also "linear" in the second argument?

Do inner products relate to norms at all?

Tell me about orthogonality.

What if I've got two vectors that are not orthogonal, but I'd like them to be?



Given: $\vec{x}, \vec{v}$

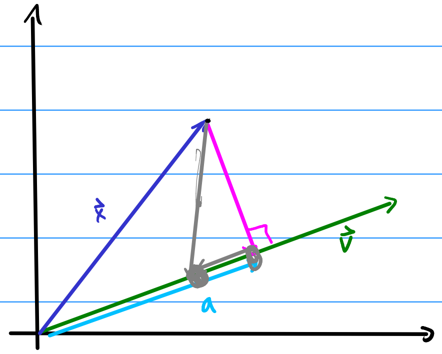
Want: $\vec{a}, \vec{b}$

with $\vec{a} = \alpha \vec{v}$

$\vec{b} \perp \vec{v}$

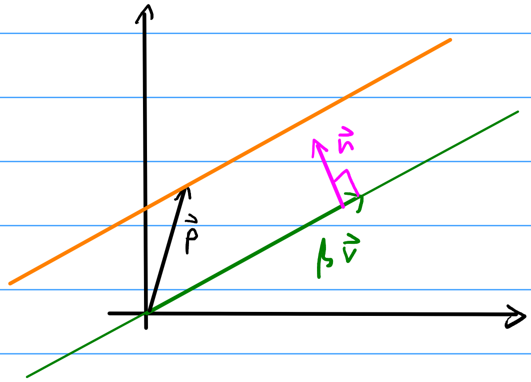
$\vec{x} = \vec{a} + \vec{b}$

In this situation, where is the closest point to $\vec{x}$ on the line $\beta\vec{v}$?



$$\vec{a} = \vec{x} - \frac{(\vec{x},\vec{v})}{(\vec{v},\vec{v})} \vec{v}$$

How does the point-normal form (of a line) work?



$$\|\vec{n}\|_2 = 1$$

Find n by picking a random vector and orthogonalizing against v.

Then multiply by 1/||n|| to achieve norm = 1.

Green line: $\vec{n} \cdot \vec{x} = 0$

Orange line: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ (the line given by $\vec{p} + \beta\vec{v}$)

$$\vec{n} \cdot \vec{x} - \underset{\vec{n} \cdot \vec{p}}{\underset{\shortparallel}{r}} = 0$$

Given a line in point-normal form $\vec{n} \cdot \vec{x} - r = 0$, an arbitrary point (with $\|\vec{n}\|_2 = 1$)

x has distance $|\vec{n} \cdot \vec{x} - r|$ from the line.

What's an orthogonal basis?

A basis whose vectors are orthogonal.

$\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$

$\vec{x}_i \perp \vec{x}_j$ if $i \neq j$

$\vec{x}_1 \perp \vec{x}_2$
$\vec{x}_2 \perp \vec{x}_3$
$\vec{x}_1 \perp \vec{x}_3$

What's an orthonormal basis (or "ONB")?

$\vec{b}_1, \dots, \vec{b}_n$

The same, but with $\|\vec{x}_i\| = 1$.

For some given vector $\vec{x}$, how do I find coefficients with respect to an ONB?

$$\vec{x} = \underbrace{(x \cdot b_1)}_{\otimes} \vec{b}_1 + \underbrace{(x \cdot b_2)}_{\otimes} \vec{b}_2 + \underbrace{(x \cdot b_3)} \vec{b}_3 + \cdots + \underbrace{(x \cdot b_n)}_{\otimes} \vec{b}_n$$

Much easier (and cheaper) than solving a linear system to find

those coefficients. The cost here is only O(n^2).

Can we build a matrix that computes those coefficients for us?

$$Q = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{pmatrix} \qquad Q^T x = \begin{pmatrix} - & b_1 & - \\ - & b_2 & - \\ & \vdots & \\ - & b_n & - \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} b_1 \cdot x \\ \vdots \\ b_n \cdot x \end{pmatrix} \begin{matrix} -\otimes \\ -\otimes \\ \\ -\otimes \end{matrix}$$

A square matrix whose columns are orthonormal is called orthogonal.

$$Q^T b_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad Q^T b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad Q^T Q = I$$

What else is true for a orthogonal matrix Q?

$$Q^T Q = I \quad \rightsquigarrow \quad Q^{-1} = Q^T$$

$$\|Qx\|_2^2 = (Qx, Qx) = x^T \cancel{Q^T Q} x = x^T x = \|x\|_2^2$$

Orthogonal matrices don't change 2-norms.

What if Q contains a few zero columns instead of orthonormal vectors?

$$\downarrow$$

Orthonormal vectors seem very useful. How can we make more than two?

`$| x+y |_2` = $\|x+y\|_2$

$$\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$$

$$\|x+y\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\vec{n} = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}}$$

$$\vec{n}\cdot\vec{x} = 3$$

$$\boxed{\vec{n}\cdot\vec{x} - 3 = 0}$$

$\vec{x} = \begin{pmatrix}3\\0\end{pmatrix}$ distance from $\vec{x}$ to line:

$$\begin{aligned} |\vec{n}\cdot\vec{x} - 3| &= \left|\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}\cdot\begin{pmatrix}3\\0\end{pmatrix} - 3\right| \\ &= \left|\underbrace{\frac{1}{\sqrt{2}}}_{.707}(-3) - 3\right| \\ &= |-2.1 - 3| \\ &= 5.1 \end{aligned}$$

So, what is the QR factorization?

If life were consistent, shouldn't this be called the QU factorization?

What is the cost of QR factorization (for an nxn matrix A)?

Does QR work for non-square matrices?

Is Q still orthogonal in a "thin" QR factorization?

If I have a "thin" QR, can I obtain a "full" QR from it?

Can QR fail?