(6) Finding the nullspace of a matrix A

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underset{\substack{\uparrow \\ \text{echelon}}}{M A^T = U} \iff A^T = M^{-1} U \iff A = U^T M^{-T}$$

$$U^T = \begin{pmatrix} \quad & 0 \end{pmatrix} \qquad N(U^T) = \text{span}\left(\underset{\substack{\uparrow \\ x}}{\{0 \cdots 0, 1, 0 \cdots 0\}}, \ldots, (0 \cdots 0\,1) \right)$$

$$A = U^T M^{-T}$$

$$Ay = 0 \iff U^T \underbrace{M^{-T} y}_{x} = 0$$

$$x = M^{-T} y$$

$$\underset{\substack{\uparrow \\ N(U^T)}}{M^T x} = \underset{\nwarrow N(A)}{y}$$

$$\rightarrow \quad N(A) = M^T N(U^T)$$

5 Orthogonality and QR

The 'linear algebra way' of talking about "angle" and "similarity" between two vectors is called "inner product". We'll define this next.

So, what is an inner product?

$$(\vec{x}, \vec{y}) \rightarrow \text{number}$$

$$\|x\| := \sqrt{(x, x)}$$

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n$$

$$\vec{x} \cdot \vec{y} = 0 \qquad \vec{x} \perp \vec{y}$$

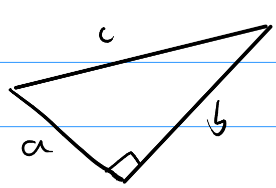
Suppose I've got x and y not orthogonal.

$$x' = x$$

$$y' = y - \frac{(x, y)}{(x, x)} x$$

$$(x', y') = 0$$

$a^2 + b^2 = c^2$



If $x \perp y$, then we have the Pythagorean theorem

$$\|x\|_2^2 + \|y\|_2^2 = \|x + y\|_2^2$$

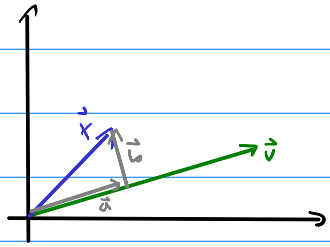
Can you give an example?

Is any old inner product also "linear" in the second argument?

Do inner products relate to norms at all?

Tell me about orthogonality.

What if I've got two vectors that are not orthogonal, but I'd like them to be?



Given: $\vec{x}, \vec{v}$

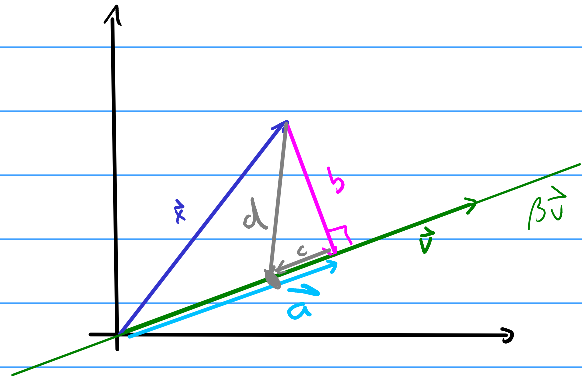
Want: $\vec{a}, \vec{b}$

with $\vec{a} = \alpha \vec{v}$

$\vec{b} \perp \vec{v}$

$\vec{x} = \vec{a} + \vec{b}$

In this situation, where is the closest point to $\vec{x}$ on the line $\beta\vec{v}$?



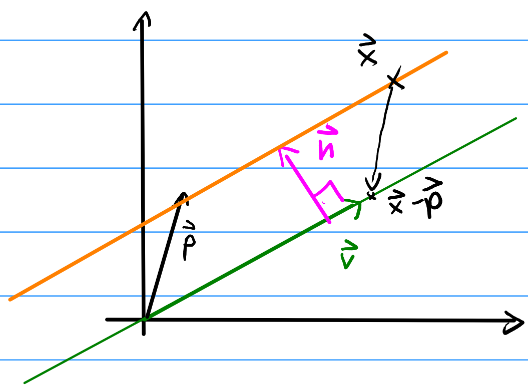
$$d^2 = b^2 + c^2$$

$$\vec{b} = \vec{x} - \frac{(\vec{x},\vec{v})}{(\vec{v},\vec{v})}\vec{v}$$

$$\vec{a} = \frac{(\vec{x},\vec{v})}{(\vec{v},\vec{v})}\vec{v}$$

$$\vec{a} + \vec{b} = \vec{x} \qquad \vec{a} = \vec{x} - \vec{b}$$

How does the point-normal form (of a line) work?



$$(\vec{n},\vec{v}) = 0 \qquad \|\vec{n}\| = 1$$

Green line: $\vec{x} \cdot \vec{n} = 0$

n is called the normal vector

Orange line: $(\vec{x} - \vec{p}, \vec{n}) = 0$

signed distance to (orange) line: $\underbrace{(x,\vec{n}) - \underbrace{(\vec{p},\vec{n})}_{r}}_{sd} = 0$

|sd| results in the distance of x to the line

The sign of sd tells us whether x is above or below the line

where "above" means "in the direction that n points"

$(\vec{x}, \vec{x}) = 0 \iff x = 0$

What's an orthogonal basis?

A basis $\vec{b}_1, \ldots, \vec{b}_n$ consisting of orthogonal vectors.

$b_1 \perp b_2$

$b_2 \perp b_3$

$b_1 \perp b_3$

$\rightarrow$ $b_i \perp b_j$ if $i \neq j$

What's an orthonormal basis (or "ONB")?

$\|b_i\|_2 = 1$

For some given vector $\vec{x}$, how do I find coefficients with respect to an ONB?

$$\vec{x} = \underbrace{(\vec{x}, \vec{b}_1)}\, \vec{b}_1 + \underbrace{(\vec{x}, \vec{b}_2)}\, \vec{b}_2 + \underbrace{(\vec{x}, \vec{b}_3)}\, \vec{b}_3 + \cdots + \underbrace{(\vec{x}, \vec{b}_n)}\, \vec{b}_n$$

$\rightarrow O(n^2)$ cost

Can we build a matrix that computes those coefficients for us?

$$Q = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix} \qquad Q^T = \begin{pmatrix} — b_1 — \\ \vdots \\ — b_n — \end{pmatrix} \qquad \begin{pmatrix} x \end{pmatrix}$$

$$Q^T x = \begin{pmatrix} — b_1 — \\ \vdots \\ — b_n — \end{pmatrix} \begin{pmatrix} b_1 \cdot x \\ \vdots \\ b_n \cdot x \end{pmatrix}$$

A square matrix with orthonormal columns is called orthogonal.

$\triangle$

$$Q^T b_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad Q^T b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad Q^T Q = I$$

What else is true for a orthogonal matrix Q?

$Q^T Q = I$ $\quad\quad$ $Q^{-1} = Q^T$

What if Q contains a few zero columns instead of orthonormal vectors?

Orthonormal vectors seem very useful. How can we make more than two?

So, what is the QR factorization?

If life were consistent, shouldn't this be called the QU factorization?

What is the cost of QR factorization (for an nxn matrix A)?

Does QR work for non-square matrices?

Is Q still orthogonal in a "thin" QR factorization?

If I have a "thin" QR, can I obtain a "full" QR from it?

Can QR fail?