What's an orthogonal basis?

What's an orthonormal basis (or "ONB")?

For some given vector $\vec{x}$, how do I find coefficients with respect to an ONB? ($\vec{b}_i$)

$$\vec{x} = (\vec{x}\cdot\vec{b}_1)\,\vec{b}_1 + (\vec{x}\cdot\vec{b}_2)\,\vec{b}_2 + (\vec{x}\cdot\vec{b}_3)\,\vec{b}_3 + \cdots + (\vec{x}\cdot\vec{b}_n)\,\vec{b}_n$$

What if $\vec{x} = \vec{b}_1$?

$$\vec{b}_1 = \underbrace{(\vec{x}\cdot\vec{b}_1)}_{1}\,\vec{b}_1 + \underbrace{(\vec{x}\cdot\vec{b}_2)}_{0}\,\vec{b}_2 + \underbrace{(\vec{x}\cdot\vec{b}_3)}_{0}\,\vec{b}_3 + \cdots + \underbrace{(\vec{x}\cdot\vec{b}_n)}_{0}\,\vec{b}_n$$

Can we build a matrix that computes those coefficients for us?

What else is true for a orthogonal matrix Q?

$$Q^T = Q^{-1} \qquad Q Q^T = Q^T Q = I$$

$$\|Qx\|_2^2 = (Qx)\cdot(Qx) = (Qx)^T Qx = x^T \cancel{Q^T Q} x = x^T x = \|x\|_2^2$$

What if Q contains a few zero columns instead of orthonormal vectors?

$$Q = \begin{pmatrix} | & | & & | & \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k & 0 \\ | & | & & | & \end{pmatrix} \qquad \vec{b}_i \cdot \vec{b}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$P = Q Q^T \neq Id$$

$$Px = Q Q^T x = Q \underbrace{\begin{pmatrix} — \vec{b}_1 — \\ — b_2 — \\ \vdots \\ — b_k — \\ 0 \quad 0 \end{pmatrix}}_{Q^T} \begin{pmatrix} | \\ \vec{x} \\ | \end{pmatrix} = Q \underbrace{\begin{pmatrix} x\cdot b_1 \\ x\cdot b_2 \\ \vdots \\ x\cdot b_k \\ 0 \end{pmatrix}}_{Q^T x}$$

$$= \begin{pmatrix} | & | & & | & \\ b_1 & b_2 & \cdots & b_k & 0 \\ | & | & & | & \end{pmatrix} \begin{pmatrix} x\cdot b_1 \\ x\cdot b_2 \\ \vdots \\ x\cdot b_k \\ 0 \end{pmatrix}$$

$$= (x\cdot b_1)\, b_1 + \cdots + (x\cdot b_k)\, b_k$$

$$P^2 = Q Q^T Q Q^T = \underbrace{\begin{pmatrix} ||| & 0 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \equiv \\ 0 \end{pmatrix} \begin{pmatrix} ||| & 0 \end{pmatrix}}_{Q^T Q} \underbrace{\begin{pmatrix} \equiv \\ 0 \end{pmatrix}}_{Q} = Q Q^T$$

$$Q^TQ = \begin{pmatrix} — b_1 — \\ \vdots \\ — b_n — \\ 0 \end{pmatrix} \begin{pmatrix} | & & | & \\ b_1 & \cdots & b_k & 0 \\ | & & | & \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}$$

Orthonormal vectors seem very useful. How can we make more than two?

	b_1	b_2	b_3
b_1			
b_2	0		
b_3	0	0	

$$\otimes \quad \|Qx\|_2 = \|x\|_2$$

$$\|Q\|_2 = \max_{\|x\|_2=1} \|Qx\|_2 \overset{\otimes}{=} \max_{\|x\|_2=1} \|x\|_2 = 1$$

Orthogonal matrices have a 2-norm of 1.

$$\text{cond}_2(Q) = \|Q\|_2 \, \|Q^{-1}\|_2 = \|Q\|_2 \, \|Q^T\|_2 = 1 \cdot 1$$

So, what is the QR factorization?

If life were consistent, shouldn't this be called the QU factorization?

What is the cost of QR factorization (for an nxn matrix A)?

Does QR work for non-square matrices?

Is Q still orthogonal in a "thin" QR factorization?

If I have a "thin" QR, can I obtain a "full" QR from it?

Can QR fail?

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Applications of QR

(1) Solve square linear systems. How?

(2) Solve tall and skinny linear systems. How?

Is there other notation for least-squares problems?

And how does QR help with least-squares problems?

So how do I solve a least-squares problem with QR?

What about the "normal equations"?

How can I use least-squares problems for fitting a linear model to data?