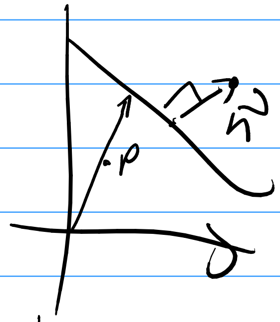
Discuss P1 on the quiz

$$\vec{x} \cdot \vec{n} - r = 0$$



$$d = \vec{x} \cdot \vec{n} - r$$

$\|n\|_2 = 1$

point $\vec{p}$ on line

$r = \vec{p} \cdot \vec{n}$.

$$\vec{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r = 2$$

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\frac{1}{\sqrt{2}}} - 2$$

$$-2$$

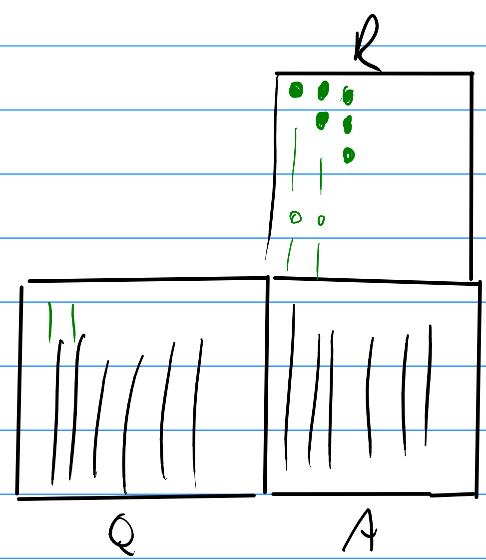
Orthonormal vectors seem very useful. How can we make more than two?

So, what is the QR factorization?

$$A = QR$$

orthogonal

upper triangular



If life were consistent, shouldn't this be called the QU factorization?

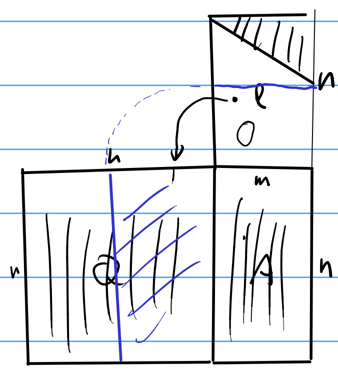
Yes.

What is the cost of QR factorization (for an nxn matrix A)?

for each column of A: $\leftarrow n$
    for each "previous" column of Q: $\leftarrow \leq n$
        Orthgonalize the column of A against the column of Q $\leftarrow O(n)$

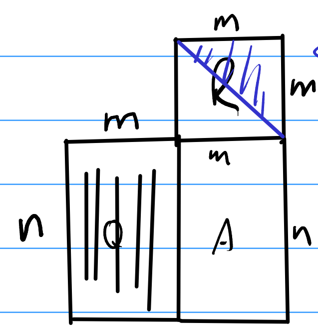
$\hookrightarrow O(n^3)$

Does QR work for non-square matrices?



A=QR

"complete QR"



square

"reduced"/"thin" QR

$$\underset{A}{\overset{n\ \ m}{(a\times c)}} = \underset{Q}{\overset{n\ \ ?}{(a\times b)}} \cdot \underset{R}{\overset{?\ \ m}{(b\times c)}}$$

Is Q still orthogonal in a "thin" QR factorization?

If I have a "thin" QR, can I obtain a "full" QR from it?

Can QR fail?

6 Applications of QR

(1) Solve square linear systems. How?

$$Ax = b \qquad A = QR$$

$$\Leftrightarrow \underbrace{QRx}_{y} = b$$

$$\rightsquigarrow Qy = b \quad | \; Q^T \cdot$$

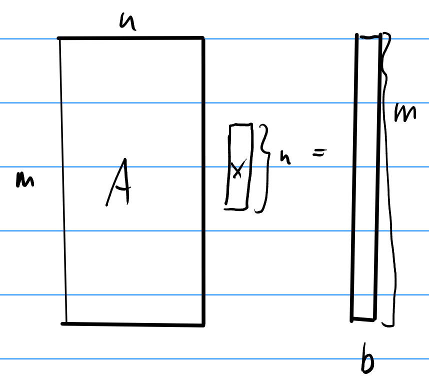
$$\rightsquigarrow \cancel{Q^TQ}\, y = Q^T b$$

$$\rightsquigarrow y = Q^T b \leftarrow O(n^2)$$

$$Rx = y \quad | \text{ bw subst } \leftarrow O(n^2)$$

works, but QR ismore expensive to obtain than LU.

(2) Solve tall and skinny linear systems. How?



In general, we won't find x so that Ax=b.

$$\|\underbrace{Ax - b}_{\text{residual}}\|_2^2 \leftarrow \text{make small}$$

$A = QR$

$$\|Q^T z\|_2^2 = \|z\|_2^2$$

$$\min \rightarrow \|Ax - b\|_2^2 = \|QRx - b\|_2^2$$

$$= \|Q^T(QRx - b)\|_2^2$$

$$= \|\underbrace{\cancel{Q^TQ}}_{I} Rx - Q^T b\|_2^2$$

$$= \|Rx - Q^T b\|_2^2$$

$$= \left\| \begin{bmatrix} R \\ 0 \end{bmatrix} x - Q^T b \right\|_2^2$$

Q orthogonal
R nonsquare

=>

"complete" QR

$$R = \begin{bmatrix} R_{upper} \\ R_{lower} \end{bmatrix} \qquad (Q^T b) = \begin{bmatrix} (Q^T b)_{upper} \\ (Q^T b)_{lower} \end{bmatrix}$$

$$\hookrightarrow = \| R_{upper} \vec{x} - (Q^T b)_{upper} \|_2^2 + \| 0 - (Q^T b)_{lower} \|_2^2$$

square

assume: invertible

solve $R_{upper} \vec{x} = (Q^T b)_{upper}$

then "green" part of residual norm is 0.

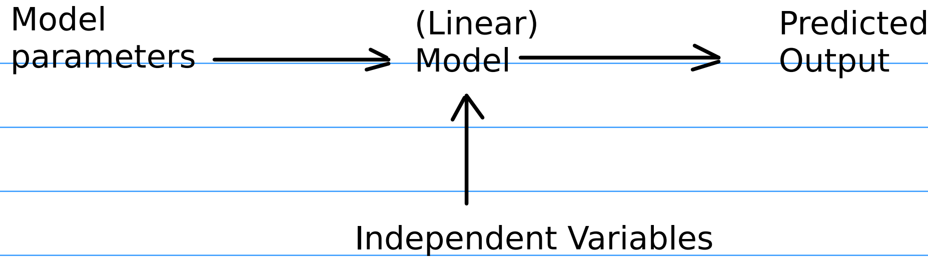
Is there other notation for least-squares problems?

And how does QR help with least-squares problems?

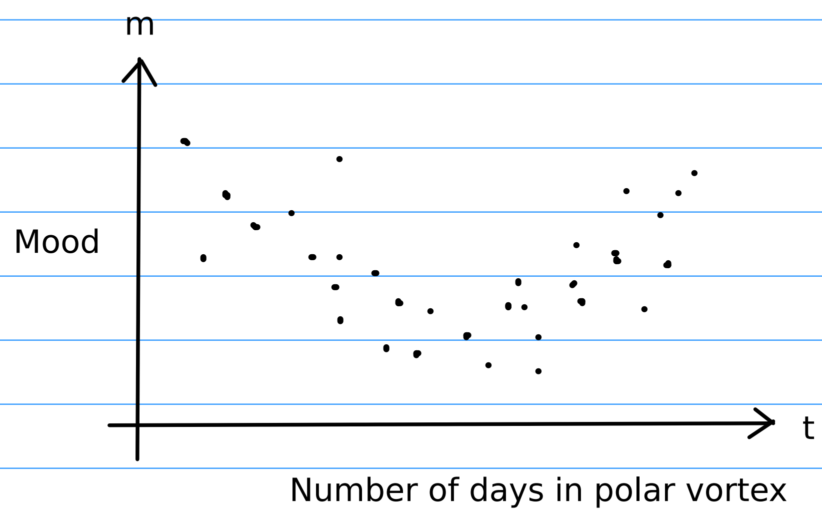
So how do I solve a least-squares problem with QR?

What about the "normal equations"?

How can I use least-squares problems for fitting a linear model to data?



Can you give an example?



How do we find the parameters then?