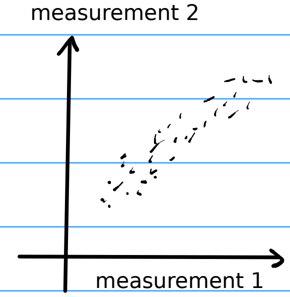
9 **Applications of the SVD**

(1) Rank-k approximation

$$A = \underbrace{\begin{pmatrix} | & | & | \\ | & | & | \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} - \\ - \\ - \end{pmatrix}}_{V^T}$$

$$\approx \sigma_1 \underbrace{\vec{u}_1 \vec{v}_1^T} + \sigma_2 \underbrace{\vec{u}_2 \vec{v}_2^T} + \cdots +$$

(4) Principal Component Analysis ("PCA")



Have: a pile of "data"

More precisely: m 'measurements' from n 'trials'

each resulting in a real number

$x_{ij} \qquad i = 1 \ldots m \qquad j = 1 \ldots n$

Data matrix: $X = (x_{ij}) = \left(\,|\,|\,|\,|\,|\,|\,| \, \right)$ ↓ i: measurements

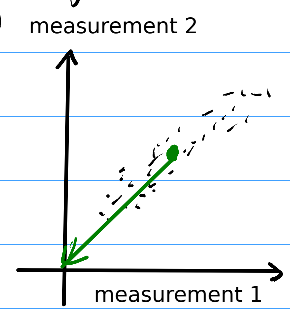
→ j: trials

Want: Underlying relationships between measurements

"If measurement i changes by amount x, then I want to be able to predict that measurement i' will change by an amount I can compute from x."

How do I compute a PCA?

(1) Compute an estimate of the mean: $u_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}$



(2) "Remove" the means: $Y_{ij} = X_{ij} - u_i$

→ Y[i,j] have mean zero

(3) Compute (an estimate of) the covariance matrix:

$$C_{i_1 i_2} = \underbrace{\frac{1}{n-1}} \sum_{j=1}^{n} Y_{i_1 j} Y_{i_2 j} \qquad C = Y \cdot Y^T$$

unbiased estimate of the cov. matrix

$(AB)_{ij} = \sum_{k} A_{ik} B_{kj}$

How do I compute a PCA? (cont'd)

(4) Diagonalize the covariance matrix (sym.pos.def.)

diagonal

$$C = U \Sigma^2 U^T$$

orthogonal

$$\Sigma^2 = U^T C U$$

$$C = \frac{1}{n-1} Y Y^T = U \Sigma \cancel{V^T V} \Sigma U^T = U \Sigma^2 U^T$$

(5) Transform the data into their "principal components"

Find a V so that $\frac{1}{\sqrt{n-1}} Y = U \Sigma V^T$

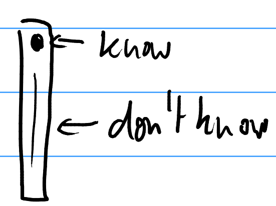
determines a linear combination of the columns of U

-> PCA can be computed by taking the SVD of

$$\frac{1}{\sqrt{n-1}} Y$$

columns of U are the "principal components"

$$\frac{1}{\sqrt{n-1}} Y \approx \sigma_1 u_1 v_1^T$$



(6) "Total" least squares

For a given matrix A, find the vector x so that

- $\|Ax\|_2$ is minimal
- $\|x\|_2 = 1$

$$A = \begin{array}{c} u_1 \quad \cdots \quad u_n \\ \begin{pmatrix} \vert & \vert & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert & \vert \end{pmatrix} \end{array} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_{n-1} & \\ & & & 0 \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$$

$\|v_n\|_2 = 1$ ✓

$$\|Av_n\|_2 = \left\| \begin{array}{c} u_1 \quad \cdots \quad u_n \\ \begin{pmatrix} \vert & \vert & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert & \vert \end{pmatrix} \end{array} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_{n-1} & \\ & & & 0 \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix} v_n \right\|_2$$

$$= \left\| \begin{array}{c} u_1 \quad \cdots \quad u_n \\ \begin{pmatrix} \vert & \vert & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert & \vert \end{pmatrix} \end{array} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_{n-1} & \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\|$$

$$= \left\| \begin{array}{c} u_1 \quad \cdots \quad u_n \\ \begin{pmatrix} \vert & \vert & \vert & \vert & \vert \\ \vert & \vert & \vert & \vert & \vert \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\|_2 = 0$$

In this case, but also in general: v_n is the solution to TLS

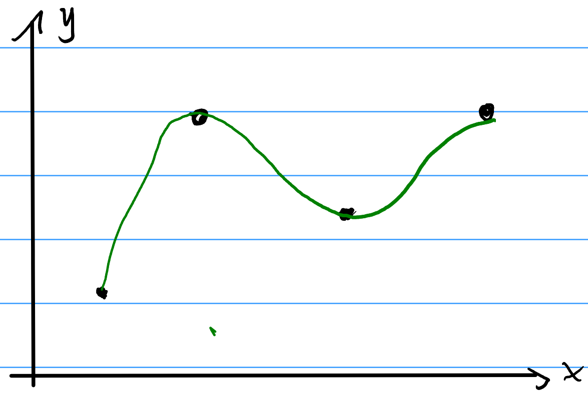
10 **Function Spaces**

What is interpolation?

$$1 \quad x \quad x^2 \quad \cdots \quad x^{n-1}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

Given a basis of functions $\varphi_1 \ldots \varphi_n$

match the data points



given

x	y
x_1	y_1
x_2	y_2
x_m	y_m

But there are lots of possible functions through those points. Be more precise.

$$V = \begin{array}{c} \begin{array}{ccc} \alpha_1 & \cdots & \alpha_n \end{array} \\ \begin{pmatrix} \varphi_1(x_1) & \cdots & \varphi_n(x_1) \\ \vdots & & \vdots \\ \varphi_1(x_n) & & \varphi_n(x_n) \end{pmatrix} \end{array}$$

$$V\vec{\alpha} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \rightarrow \quad \alpha_1\varphi_1(x) + \cdots + \alpha_n\varphi_n(x) = f(x)$$

$$\alpha_1\varphi_1'(x) + \cdots + \alpha_n\varphi_n'(x) = f'(x)$$

$$V \cdot \begin{pmatrix} \text{basis} \\ \text{coeffs} \end{pmatrix} = \begin{pmatrix} \text{function} \\ \text{values} \end{pmatrix} \rightsquigarrow \quad V^{-1}\begin{pmatrix} \text{function} \\ \text{values} \end{pmatrix} = \begin{pmatrix} \text{basis} \\ \text{coeffs} \end{pmatrix}$$

$$V' = \begin{pmatrix} \varphi_1'(x_1) & \cdots & \varphi_n'(y_1) \\ \vdots & & \vdots \\ \varphi_1'(x_n) & & \varphi_n'(x_n) \end{pmatrix}$$

$$V' \begin{pmatrix} \text{coeffs} \end{pmatrix} = \begin{pmatrix} \text{function values of the derivative} \end{pmatrix}$$

$$\underbrace{V' \; \underbrace{V^{-1} \begin{pmatrix} \text{point values} \end{pmatrix}}_{\text{coeffs}}}_{\text{function values of the derivative}}$$

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in-class:

$\varphi_0(x) = 1$

$\varphi_1(x) = x$

$\varphi_2(x) = x^2$

$\varphi_3(x) = x^3$

$$V = \begin{pmatrix} \varphi_1(x_1) & \cdots & \varphi_n(y_1) \\ \vdots & & \vdots \\ \varphi_1(x_n) & & \varphi_n(x_n) \end{pmatrix}$$

$$V\vec{\alpha} = \vec{y}$$