So how would I use calculus on an interpolant?

$f(x_i) = \tilde{f}(x_i) \quad i=1 \ldots n$

Have: interpolant $\hat{f}(x) = \alpha_1 \varphi_1(x) + \cdots + \alpha_n \varphi_n(x)$

Want: derivative

$$\vec{\alpha} = V^{-1} \vec{f}$$

$$\hat{f}'(x) = \alpha_1 \varphi_1'(x) + \cdots + \alpha_n \varphi_n'(x)$$

$$V' = \begin{pmatrix} \varphi_1'(x_1) & \cdots & \varphi_n'(x_1) \\ (x_n) & & (x_n) \end{pmatrix}$$

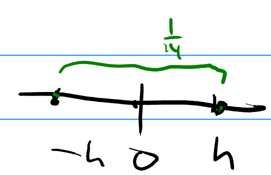
$$\vec{\hat{f}'} = \underbrace{V' V^{-1}}\vec{f}$$

So how would I use calculus on an interpolant? (cont'd)

Give a matrix that takes two derivatives.

What is the observed behavior of the error when taking a derivative?

What do the entries of the differentiation matrix mean?

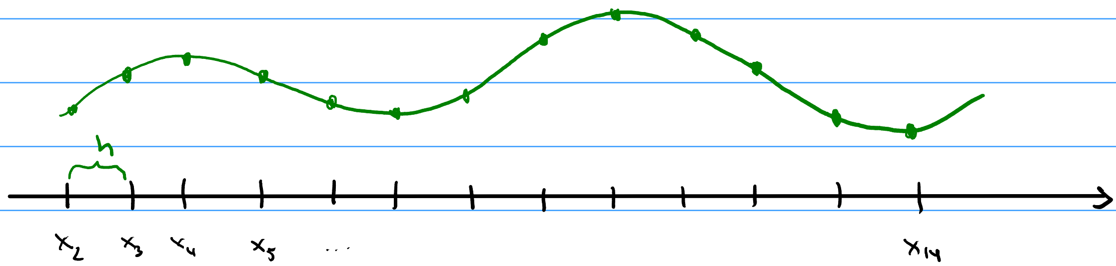


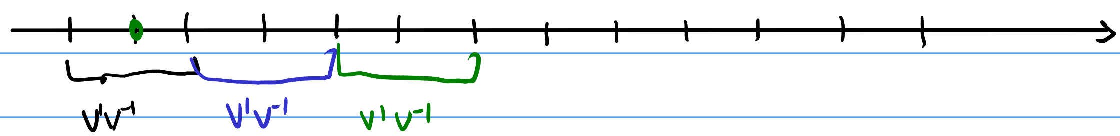
do not care

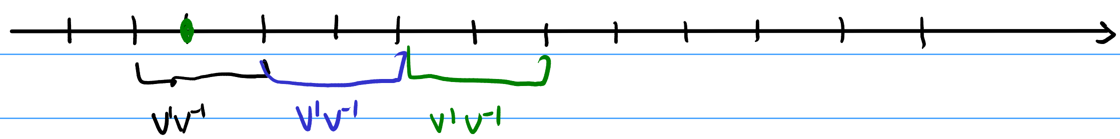
$$D = V' V^{-1} = \begin{pmatrix} & & \\ -\frac{1}{2h} & 0 & \frac{1}{2h} \\ & & \end{pmatrix} \qquad D\vec{f} = V' V^{-1} \underbrace{\begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}}_{\vec{f}} = \begin{pmatrix} \\ f(x_0)\cdot\left(-\frac{1}{2h}\right) + f(x_2)\cdot\left(\frac{1}{2h}\right) \\ \\ \end{pmatrix}$$

$$f'(x_1) \approx f(x_0)\cdot\left(-\frac{1}{2h}\right) + f(x_2)\cdot\left(\frac{1}{2h}\right) = \frac{f(x+h) - f(x-h)}{2h}$$

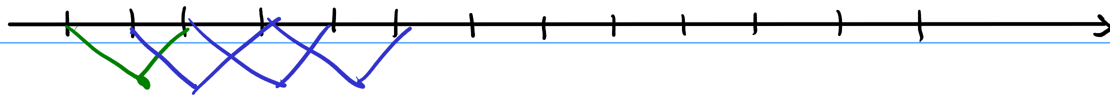
Can we simplify the proccess for equispaced x-coordinates?



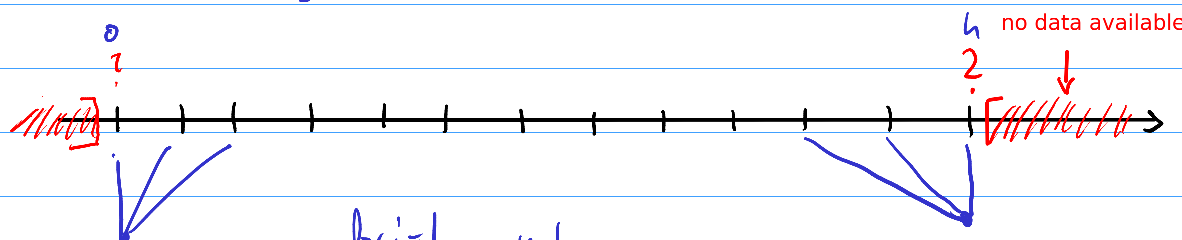




How would we use translation invariance with centered differences?



What do we do at the edges?



for $i=1,\dots,n-1$

$$f'(x_i) \approx \left(f(x_{i+1}) - f(x_{i-1})\right)/2h$$

$$\max_{x\in[a,b]} \left| f'(x) - \hat{f}'(x) \right| \le C\cdot h^n$$ for max poly degree n

Interpolation: $\max \left| f(x) - \tilde{f}(x) \right| \le C\cdot h^{n+1}$

How accurate are finite difference formulas?

$$(x^n)' = n\cdot x^{n-1}$$

$$\left| f'(x) - \hat{f}'(x) \right| \le C\cdot h^n$$

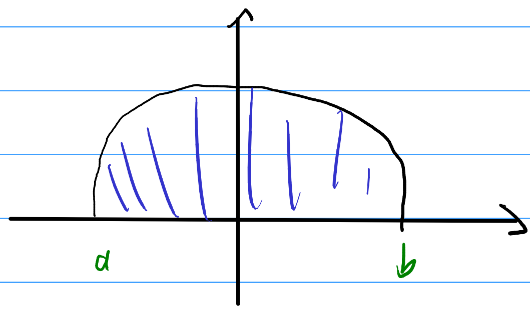
h^n f'

h^{n+1} f

h^{n+2} $\int f$

Can anything else go wrong when we numerically compute derivatives?

Can we use a similar process to compute (approximate) integrals?



Quadrature $\uparrow$

Have: interpolant $\tilde{f}(x) = \alpha_1 \varphi_1(x) + \cdots + \alpha_n \varphi_n(x)$

$$\int_a^b \tilde{f}(x)\,dx = \alpha_1 \underbrace{\int_a^b \varphi_1(x)\,dx}_{d_1} + \cdots + \alpha_n \underbrace{\int_a^b \varphi_n(x)\,dx}_{d_n}$$

$$\vec{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \qquad \vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \\ \alpha_n \end{pmatrix}$$

$$\vec{f} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

$$\vec{\alpha} = V^{-1}\vec{f}$$

$$\int_a^b \tilde{f}(x)\,dx = \vec{d}^T\vec{\alpha} = d_1\alpha_1 + \cdots + d_n\alpha_n$$

$$= \vec{d}^T (V^{-1}\vec{f})$$

$$= (V^{-1}\vec{f})^T \vec{d}$$

$$= \vec{f}^T V^{-T} \vec{d}$$

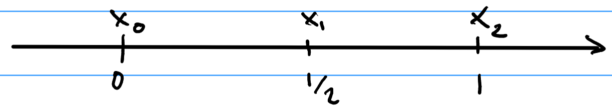
$$= \vec{f}^T \underbrace{(V^{-T}\vec{d})}_{\substack{\text{quadrature weights} \\ \vec{w}}}$$

$$= w_1 f_1 + \cdots + w_n f_n$$

$$= w_1 f(x_1) + \cdots + w_n f(x_n)$$

So, once I know my nodes and my weights, what does quadrature look like?

Can you do a full example?



$f(x_0) = 2 \qquad f(x_1) = 0 \qquad f(x_2) = 3$

1 Find coefficients $\quad \vec{\alpha} = V^{-1}\begin{pmatrix}2\\0\\3\end{pmatrix}$

2 Compute integrals

$$\int_0^1 1 \, dx = 1 = d_1$$
$$\int_0^1 x \, dx = 1/2 = d_2$$
$$\int_0^1 x^2 \, dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3} = d_3$$

3 Combine it all together

$$\vec{d} \cdot V^{-1}\begin{pmatrix}2\\0\\3\end{pmatrix}$$

$$\int_0^1 \tilde{f}(x)\, dx = \begin{pmatrix}.166\\.66\\.166\end{pmatrix} \cdot \begin{pmatrix}f(0)\\f(1/2)\\f(1)\end{pmatrix}$$

Simpson's rule

What does Simpson's Rule look like on [0, 1/2]?

What does Simpson's Rule look like on [5, 6]?

How accurate is Simpson's rule?