Constrained Optimization

Modify the problem statement of optimization to accommodate a constraint.

Objective function $f : \mathbb{R}^n \to \mathbb{R}$

$\to \min_x f(x)$

Constraint function $g : \mathbb{R}^n \to \mathbb{R}^m$

$g(x) = 0$

What does a solution/minimum x^* of this problem look like?

I.e. what are some <u>necessary</u> conditions on x^* ?

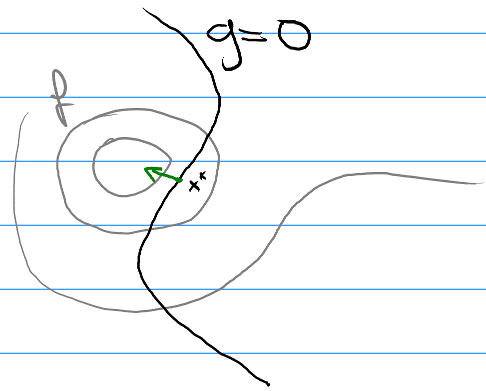
$g(x^*) = 0$

All descent directions for f at x* must cause the constraints

to be violated.

$-\nabla f(x^*) \in \text{rowspace } J_g \qquad J_g : {}_m\overset{n}{\boxed{\equiv}}$

$-\nabla f(x) = J_g^T \vec{\lambda} \qquad \text{for some } \lambda$



$$-\nabla f = J_g^T \lambda$$

Miracle: Reduce constrained to un-constrained optimization.

Define a new function of more unknowns: x and λ, $\lambda \in \mathbb{R}^m$

$$\mathcal{L}(x, \lambda) := f(x) + g(x)^T \lambda$$

What are the necessary conditions for an un-constrained minimum of $\mathcal{L}$?

$$\nabla \mathcal{L} = 0 = \begin{pmatrix} \nabla_x \mathcal{L} \\ \nabla_\lambda \mathcal{L} \end{pmatrix} = \begin{pmatrix} \nabla f(x) + J_g^T(x)\lambda \\ g(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using Newton's method on $\mathcal{L}$ gets a new name:

Sequential Quadratic Programming

Can you do an example?

Minimize $\underbrace{(x-2)^2}_{f(x,y)=} + 2(y-1)^2$ subject to $x+4y=3$

$g(x)=0$

$g(x,y)=x+4y-3$

Unc. min $x=2 \quad y=1$ ✗

$$\mathcal{L}(x,y,\lambda) = f(x,y) + g(x)^T\lambda$$

$$= (x-2)^2 + 2(y-1)^2 + (x+4y-3)\cdot\lambda$$