Are elimination matrices invertible?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad (\cdot 4) \quad -7$$

$$M_1' = \begin{pmatrix} 1 & & \\ -4 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \qquad (M_1')^{-1} = \begin{pmatrix} 1 & & \\ 4 & 1 & \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_1'' = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -7 & 0 & 1 \end{pmatrix} \qquad (M_1'')^{-1} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 7 & 0 & 1 \end{pmatrix}$$

$$M_1 = M_1' M_1'' = \begin{pmatrix} 1 & & \\ -4 & 1 & \\ -7 & & 1 \end{pmatrix} \rightarrow M_1^{-1} = \begin{pmatrix} 1 & & \\ 4 & 1 & \\ 7 & 0 & 1 \end{pmatrix}$$

With enough elimination matrices, we should be able to arrive at REF...

$$M_2 M_1 A = U \qquad \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \leftarrow \text{upper triangular}$$

What happens if we combine many elimination matrices like that?

We could rearrange that relationship to get a factorization of A!

LU

$$A = \underbrace{\triangle}\ \underbrace{\nabla}$$

$$M_2 M_1 A = U \quad \Big|\ M_2^{-1}\cdot$$

$$M_1 A = M_2^{-1} U \quad \Big|\ M_1^{-1}\cdot$$

$$A = \underbrace{M_1^{-1} M_2^{-1}}_{L} \underbrace{U}_{\nabla}$$

$$A = LU$$

So is LU/Gaussian elimination bulletproof?

$$\begin{pmatrix} a & \\ b & \end{pmatrix} \quad -\frac{b}{a} \; ? \; \approx 0$$