$Ax \simeq b$

A has nullspace: $n \neq 0$ $An = 0$

Have solution x that minimizes $\|Ax-b\|_2$

$$A(x+\alpha n) = Ax + \alpha An = Ax$$

$\rightarrow$ extra condition: also minimize $\|x\|_2$

$$A = U\Sigma V^T$$

$$\|Ax-b\|_2^2 = \|U\Sigma V^T x - b\|_2^2$$

$$= \|U^T(U\Sigma V^T x - b)\|_2^2$$

$$= \|\Sigma V^T x - U^T b\|_2^2 \leftarrow y = V^T x$$

$$= \|\Sigma y - U^T b\|_2^2$$

$$= \left\| \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_k & 0 \\ 0 & & & 0 \end{pmatrix} y - U^T b \right\|_2^2 \quad \sigma_1, \ldots, \sigma_n \neq 0.$$

$\hookrightarrow$ $y_1 = (U^T b)_1 / \sigma_1$

$\vdots$

$$\Sigma^+ = \begin{pmatrix} 1/\sigma_1 & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_k} & 0 \\ 0 & & & 0 \end{pmatrix}$$

$$y = \Sigma^+ U^T b$$

$V^T x = \Sigma^+ U^T b \quad | \; V \cdot$

$$\boxed{x = \underbrace{V \Sigma^+ U^T}_{A^+} b}$$

A^+ pseudo inverse (of A)

X solves the least squares problem,
and it also minimizes the 2-norm of x