(4) Applications of LU

(1) Solve linear equations. How?

$PA = LU \quad \rightsquigarrow \quad \Leftrightarrow PAx = Pb$

$\Leftrightarrow L\underbrace{Ux}_{\text{"}y\text{"}} = Pb$

(1) Solve $Ly = Pb$

(2) Solve $Ux = y$

Isn't this complicated or expensive?

(No: The factorization itself is cheap--and reusable.)

(2) Solve a matrix equation. How?

known

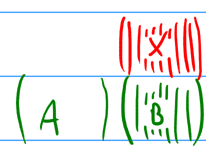
Given: $AX = B$

unknown

Simplifying assumption:

A, X, B are square and have same size.

We can solve this column-by-column:



No different than solving lots of linear systems with the same A

and lots of different right-hand side vectors b. Can reuse L and U.

<u>Example:</u> $AX = I$ (I is the identity matrix.)

$\rightsquigarrow$ Solved by $X = A^{-1}$.

$\rightsquigarrow$ Can find inverse A^{-1} using LU

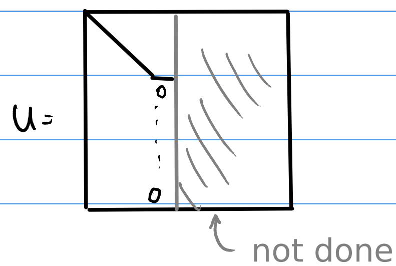
Computational Cost: $\overbrace{O(n^3)}^{\text{LU}} + n \cdot \overbrace{O(n^2)}^{\text{BW+FW subst}} = O(n^3)$

(3) Find row echelon form

The factor U in pivoted LU looks like it is in upper echelon form, and most of the time it is... but this is not guaranteed.

For example, U can contain linearly dependent rows.

Demo: LU and upper echelon form, Part I



If you hit a column of all zeros, then to achieve echelon form, you would need to "move right" (and just keep eliminating in the same row). Then our pivot/elimination split trick no longer works, and L is no longer lower triangular!

(Pivoting is the problem here!)

If you are wondering: The details of why this breaks will not be on the exam.

But: We can still use the same process as pivoted LU to compute an invertible matrix M so that

$$MA = U$$

so that U is in upper echelon form. But M cannot easily be factored into elimination and permutation matrices--and thus not easily inverted!

Nonetheless, we can obtain the "echelon factorization":

$$A = M^{-1}U$$

Demo: LU and upper echelon form, Part II

(4) Find the basis of a span. How?

Given: $\vec{x}_1, \dots, \vec{x}_n$ linearly dependent

Want: $\vec{y}_1, \dots, \vec{y}_k \in \operatorname{span}\{ \vec{x}_1, \dots, \vec{x}_n \} = V$ linearly independent

Define $A = \begin{pmatrix} — x_1 — \\ — x_n — \end{pmatrix}$

Obtain $MA = U$ in echelon form

Non-zero rows of U form a basis of V.

(5) Find the determinant of a matrix. How?

$PA = LU$

$\rightsquigarrow \det(P) \det(A) = \det(L) \det(U)$

$\det(P) \to \pm 1$

$\det(L) \to 1$

$\det(U) \to$ product of diagonal entries

(5) We'd like to find the rank* of a matrix. Is that possible using a computer?

Two randomly vectors almost surely do no not point in the same direction.

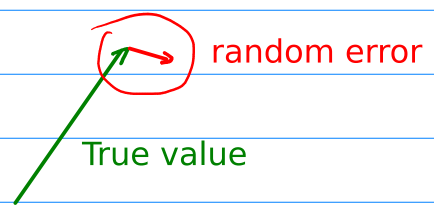
Two random vectors are almost surely <u>not</u> linearly dependent.

Computers do not represent numbers exactly. (in floating point)

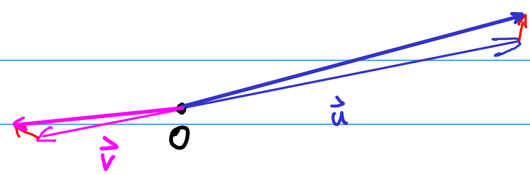
Every floating point number: $\underbrace{3.14159}_{\text{good digits}}\underbrace{777777}_{\text{junk}}$

Model that as: True value + (small?) "<u>random</u>" error

In a vector:



Suppose we would like to test two inexact vectors for linear dependence.



True: $\vec{u} = \alpha \vec{v}$ (linearly dependent)

Computed: $\vec{u} \neq \alpha \vec{v}$ (not linearly dependent)

<u>Lesson:</u> We cannot hope for exact equality on a computer.
Instead, we must define some sort of tolerance.

*rank: Number of linearly independent rows/columns

Suppose we take that into account. How would we compute the rank?

Just compute echelon form. In exact arithmetic,
"missing" row rank would appear as rows of zeros.

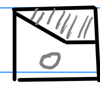
On a computer, we cannot hope for exact zeros.

Demo: Computing the Rank

Lesson: To find the rank computationally, we must specify a threshold on (for example) the minimum norm of an echelon form row.

(6) Finding the nullspace of a matrix A

Echelon factorization of A is not much help:

$$A = M^{-1} U = M^{-1}$$ 

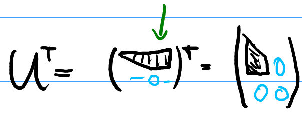
nullspace not obvious

Idea: Start with echelon factorization of A^T:

$(M^{-1})^T = (M^T)^{-1}$

$$A^T = M^{-1}U \iff A = U^T M^{-T}$$

$N(U^T) = ?$



echelon form

$$\rightsquigarrow N(U^T) = \{ [0,0,0,\dots,0,1,0,\dots,0], \; \vdots \; [0,0,0,\dots \dots,0,1] \}$$

because these vectors "hit" the zero columns in U^T.

So we know a few vectors $\vec{x}$ so that $U^T \vec{x} = \vec{0}$.

But $A = U^T M^{-T}$

We're looking for $\vec{y}$ so that $M^{-T}\vec{y} = \vec{x}$. (for each of our $\vec{x}$)

Easy: $\vec{y} = M^T \vec{x}$.

$\rightsquigarrow N(A) = M^T N(U^T)$

Demo: Computing the Nullspace