CS 357: Numerical Methods

Lecture 12:
QR Decomposition
Least Squares

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QR review

$\underline{\underline{A}} = QR$

- Reduced QR
- Let A by m by n and A=QR
  - What is the shape of Q? $m \times n$
  - What is the shape of R? $n \times n$
- What are they for Full QR?

$$\underset{m \times n}{A} = \underset{m \times m}{Q} \underset{m \times n}{\begin{pmatrix} R \\ 0 \end{pmatrix}}$$

QR review

- What is special about Q? orthonormal

  columns unitized
  orthogonal to eachother

- What space do the q1,…,qn columns span?

  basis for range (A)
  = span of
  columns of
  A

Solving Linear Systems with QR

- Ax=b for A an~~d~~ n by n matrix

- If A=QR, how can we solve for x?

$$Q\underbrace{Rx}_{y} = b$$

$$Qy = b$$

$$Q^T Q y = Q^T b$$

$$\boxed{y = Q^T b}$$

$$\underbrace{Rx = y = Q^T b}_{\text{solve w/ back substitution}}$$

- What is the computational cost?

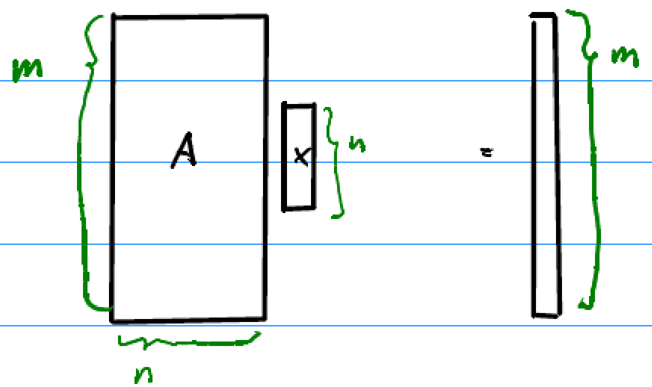
$QR \to O(n^3)$

$y = Q^T b \to O(n^2)$

solve $\to O(n^2)$

Least Squares with QR

- Tall and skinny systems



Tall and skinny: $m > n$

More equations than unknowns

- No exact solution….

“Best” solution

- Find some **x** so that **Ax** is as close to **b** as we can get
- Take the difference…and measure the magnitude using a norm
- We want to minimize $||Ax - b||_2^2$
- The **residual** is $r = Ax - b$
- We are minimizing the sum of squares of the residual
- This process is called a ***least squares problem***

Notation

- Lots of different notation that means the same thing
  - Find x so that $||Ax - b||_2^2$ is minimized
  - $x = \underline{argmin_x}||Ax - b||_2^2$
  - $Ax \cong b$

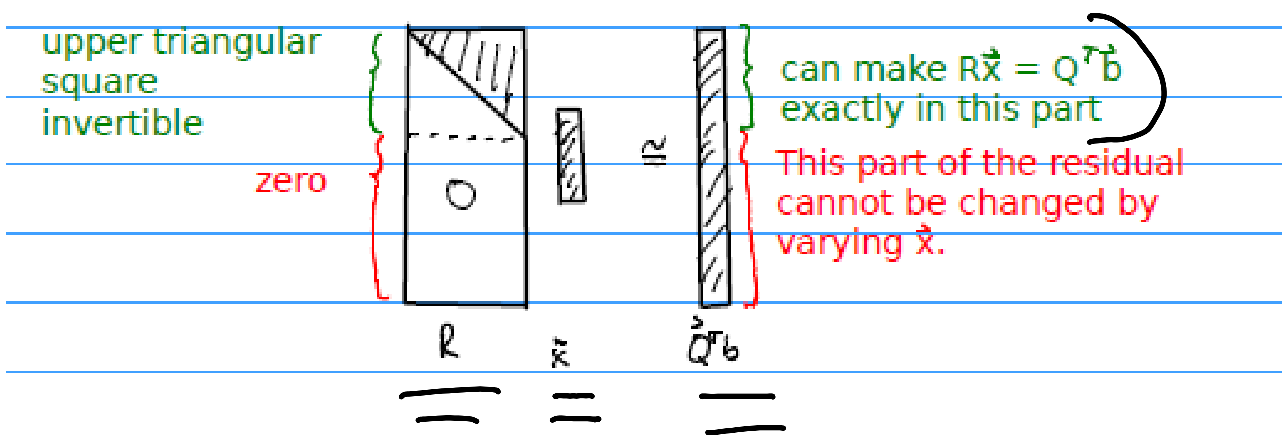
QR and Least Squares

$A = QR$

$$\underbrace{\|Ax - b\|} = \|QRx - b\|$$

$$= \|Q^T(QRx - b)\|$$

$$= \|\underbrace{Rx - Q^T b}\|$$

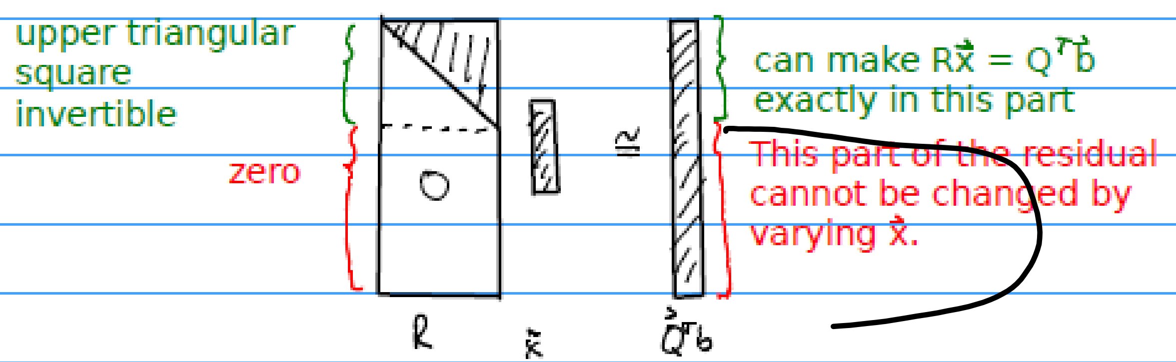


QR and Least Squares

$R_{top} x = Q^T b$ solve

That minimizes $Rx - Q^T b$

norm of residual $\|Ax - b\| = \|(Q^T b)_{bottom}\|$



The Normal Equations

$Ax \cong b$

- Another way of solving Least Squares

$A^T A x = A^T b$

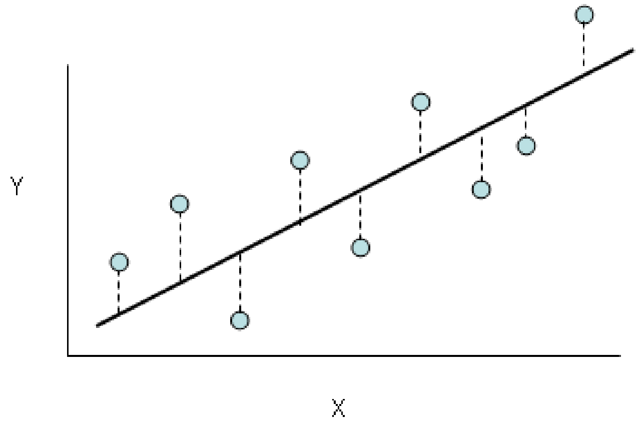
$\text{condition}\#(A^T A) = (\text{condition}\#(A))^2$

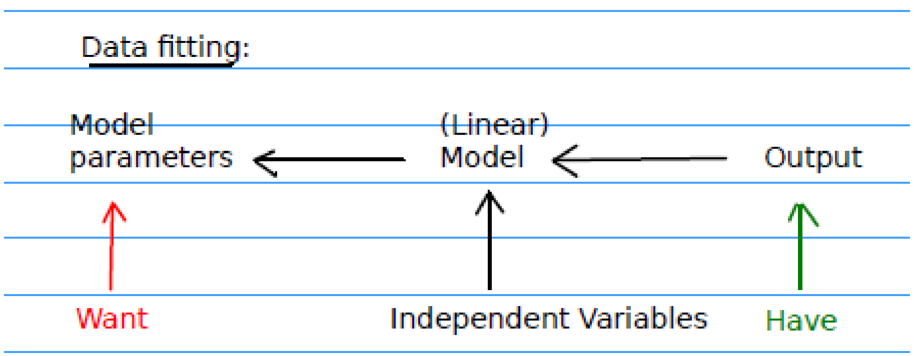
Question….

- Why is it called “linear least squares”

Data Fitting

- Lots of interesting problems lack an exact solution….
- Fit a function to a set of points….





Data Fitting

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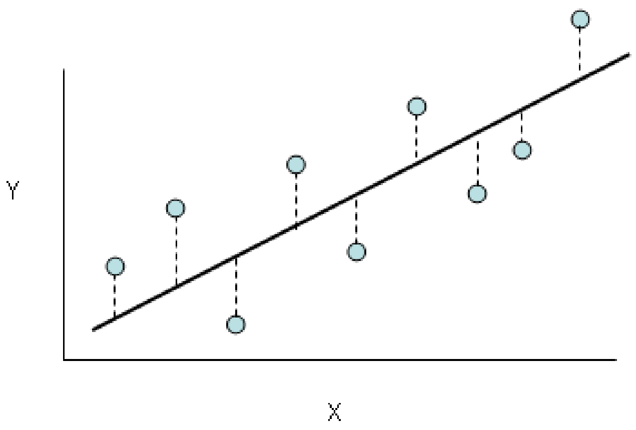

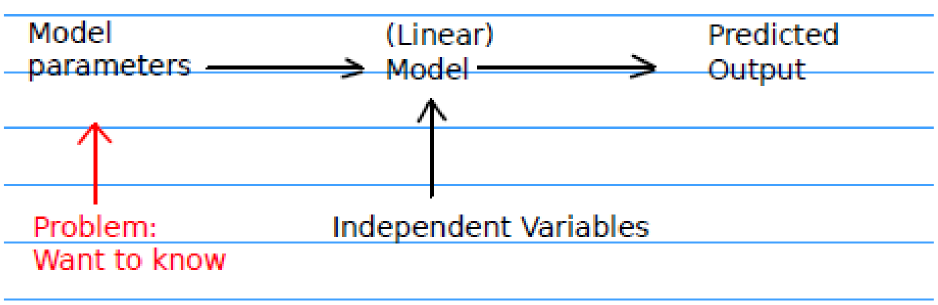

Fitting a line

m sample points $p_i = (t_i, y_i)$

$$y_i = x_1 t_i + x_0$$

$$m \overset{A}{\underset{2}{\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ & \vdots \\ 1 & t_m \end{bmatrix}}} \overset{x}{\underset{2}{\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}}} = \overset{b}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}$$

Fitting a curve

$$f(t) = \underline{\underline{x_2}}\, t^2 + \underline{\underline{x_1}}\, t + \underline{\underline{x_0}}$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ & \vdots & \\ 1 & t_m & t_m^2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_m \end{bmatrix}$$

"linear" least squares

$\longrightarrow$ x_i are linear

Fitting a curve