CS 357: Numerical Methods

Lecture 13: Eigenvalues and Eigenvectors

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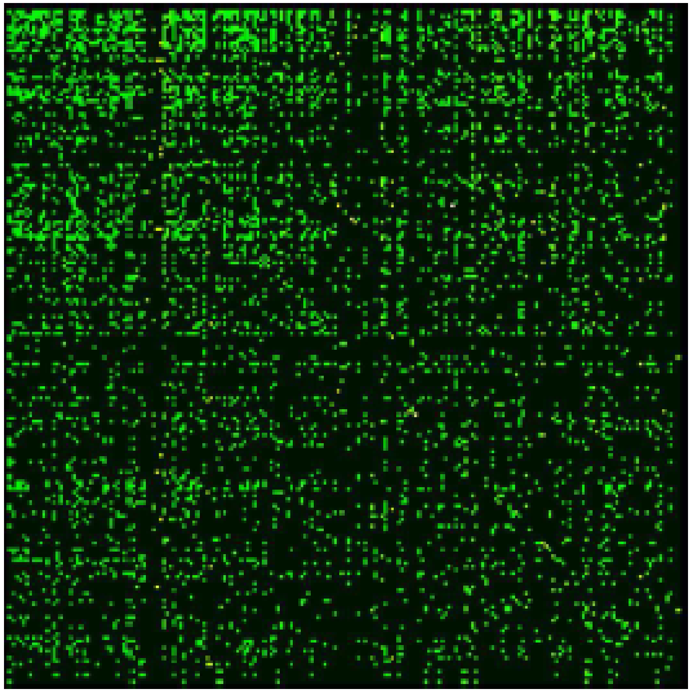
Eigenthings



Eigenvectors and eigenvalues

Reveal action and geometry

Lots of applications

Engineering design
    make sure your bridge won't
    spontaneously fall down

Data analysis
    Google and PageRank

Eigenvectors

Given matrix A: which vectors $\mathbf{r}$ mapped to a multiple of itself?

$$A\mathbf{r} = \lambda\mathbf{r} \qquad \lambda \in \mathbb{R}$$

Disregard the "trivial solution" $\mathbf{r} = \mathbf{0}$

In 2D: at most two directions
Symmetric matrices: directions orthogonal (more on that later)

Fixed directions called the eigenvectors
— from the German word "*eigen*" meaning special or proper
Factor λ called its eigenvalue
Key to understanding geometry of a matrix

Finding Eigenvalues by Hand

How to find the eigenvalues of a 2×2 matrix A

$$A\mathbf{r} = \lambda\mathbf{r} = \lambda I\mathbf{r}$$

$$[A - \lambda I]\mathbf{r} = \mathbf{0}$$

Matrix $[A - \lambda I]$ maps a nonzero vector $\mathbf{r}$ to the zero vector
$\Rightarrow [A - \lambda I]$ rank deficient matrix $\Rightarrow$

$$p(\lambda) = \det[A - \lambda I] = 0$$

Characteristic equation: polynomial equation in λ
— 2D: characteristic equation is quadratic
$p(\lambda)$ called the characteristic polynomial

Finding Eigenvalues by Hand

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$p(\lambda) = \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 1$$

Recall quadratic equation:
$a\lambda^2 + b\lambda + c = 0$ has the solutions

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Finding Eigenvalues by Hand

Example continued

Find $\mathbf{r}_1$ and $\mathbf{r}_2$ corresponding to $\lambda_1 = 3$ and $\lambda_2 = 1$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \mathbf{r}_1 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{r}_1 = \mathbf{0}$$

Homogeneous system and rank 1 matrix

$\Rightarrow$ infinitely many solutions

Forward elimination results in

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{r}_1 = \mathbf{0}$$

Assign $r_{2,1} = 1$, then $\mathbf{r}_1 = c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Finding Eigenvalues by Hand

Next: $\lambda_2 = 1$, find $\mathbf{r}_2$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \mathbf{r}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{r}_2 = \mathbf{0}$$

$$\mathbf{r}_2 = c \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Recheck Figure: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not stretched – it is mapped to itself

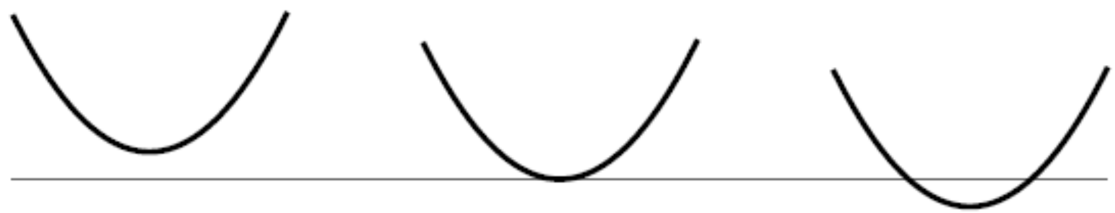
Often eigenvectors normalized for degree of uniqueness

$$\mathbf{r}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{r}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Dominant eigenvector: eigenvector corresponding to dominant eigenvalue

The Geometry of Eigenvectors

Quadratic polynomials have either no, one, or two *real* zeroes



If there are no zeroes: then A has no fixed directions

Example: rotations — rotate every vector; no direction unchanged

Rotation by -90°

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Characteristic equation

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 + 1 = 0$$

Three Ways to Say the Same Thing

$Ax=\lambda x$

$(A-\lambda I)x=0$

$x \in N(A-\lambda I)$

Matrices with Easily Found Eigenthings

- Triangular Matrices
  - Eigenvalues are the diagonal entries
  - To find eigenvectors, find $N(A-\lambda I)$

Example

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \lambda_i = 4,\ 3,\ 2,\ 1$$

Starting with $\lambda_1 = 4$:

$$\begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \mathbf{r}_1 = \mathbf{0} \quad \Rightarrow \quad \mathbf{r}_1 = \begin{bmatrix} 1/3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Repeating for all eigenvalues

$$\mathbf{r}_2 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{r}_3 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1 \end{bmatrix} \quad \mathbf{r}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and check: } A\mathbf{r}_i = \lambda_i \mathbf{r}_i$$

Finding Eigenvalues for Larger n

General $n \times n$ matrix has a degree n characteristic polynomial

$$p(\lambda) = \det[A - \lambda I] = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdot \ldots \cdot (\lambda_n - \lambda)$$

Let $\lambda = 0$ then $p(0) = \det A = \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n$

Finding zeroes of n^{th} degree polynomial nontrivial
— Use iterative method to find dominant eigenvalue (see next Section)
— Symmetric matrices always have real eigenvalues
— A and A^{T} have the same eigenvalues
— A is invertible and has eigenvalues λ_i, then A^{-1} has eigenvalues $1/\lambda_i$

Eigenvalues and Changes to A

- Scaling $\beta A x = \beta \lambda x$
- Shift $(A - \sigma I)x = Ax - \sigma \mathrm{x} = (\lambda - \sigma)\mathrm{x}$
- Matrix Power $A^k x = \lambda^k x$
- Inverse $A^{-1} x = \frac{1}{\lambda} x$

Similarity Transform

- Previous transforms left eigenvectors same, changed eigenvalues
- Can we change eigenvectors and retain eigenvalues?
- Let T be an invertible matrix. A *similarity transform* is

$$T^{-1}AT$$

- Let $y = T^{-1}x$

then $(T^{-1}AT)y = T^{-1}ATT^{-1}x = T^{-1}Ax = \lambda T^{-1}x = \lambda y$

Does every Square Matrix have n Linearly Independent Eigenvectors

- If A has n linearly independent eigenvectors xi we can create matrix X such that

$$X = [x1 \quad \ldots \quad xn]$$

$$AX = X\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = \mathrm{XD}$$

$$X^{-1}AX = D$$

- A is called ***diagonalizable***
- ***Not all matrices are diagonalizable***

Rayleigh Quotient

- We can estimate an eigenvalue if we know an eigenvector x
- Rayleigh quotient $$\frac{x \cdot Ax}{x \cdot x} = \frac{x \cdot \lambda x}{x \cdot x} = \lambda$$

The Power Method

- A is an n x n diagonalizable matrix, x an eigenavector
- It follows that $A^i x = \lambda^i x$
- This can be used to find the eigenvector x

The Power Method

Given A, and n x n diagonalizable matrix
Pick an arbitrary vector x_1

$x_k = Ax_{k-1}$

Keep going until the ratio $\dfrac{\|x_k\|}{\|x_{k-1}\|}$ stops changing much

Why does this work?

Why does this work?

The Power Method

What does $\dfrac{\|x_k\|}{\|x_{k-1}\|}$ converge to?

The Power Method

- If λ is large or small there could be numerical problems
  - What kinds?

- Can it converge if first guess is perpendicular to the eigenvector?
  - Yes…round off error means it will converge slowly…

- If $|\lambda_1| \approx |\lambda_2|$ method may not converge
- Method limited to symmetric matrices with a dominant eigenvalue

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \lambda_1 = 3 \qquad \lambda_2 = 1$$

$$A_2 = \begin{bmatrix} 2 & 0.1 \\ 0.1 & 2 \end{bmatrix} \qquad \lambda_1 = 2.1 \qquad \lambda_2 = 1.9$$

$$A_3 = \begin{bmatrix} 2 & -0.1 \\ 0.1 & 2 \end{bmatrix} \qquad \lambda_1 = 2 + 0.1i \quad \lambda_2 = 2 - 0.1i$$

$$\mathbf{r}^{(1)} = \begin{bmatrix} 1.5 \\ -0.1 \end{bmatrix} \quad \infty\text{-norm scaled} \quad \Rightarrow \quad \mathbf{r}^{(1)} = \begin{bmatrix} 1 \\ -0.066667 \end{bmatrix}$$

A_1: symmetric and λ_1 separated from λ_2
$\Rightarrow$ rapid convergence in 11 iterations — Estimate: $\lambda = 2.99998$

A_2: symmetric but λ_1 close to λ_2
$\Rightarrow$ convergence slower 41 iterations — Estimate: $\lambda = 2.09549$

A_3: rotation matrix (not symmetric) and complex eigenvalues
$\Rightarrow$ no convergence.

Normalized Power Iteration

What does this converge to?

$$x_k = \frac{Ax_{k-1}}{||Ax_{k-1}||}$$

Why would anyone do that?

Inverse Iteration

$$x_k = \frac{A^{-1}x_{k-1}}{\|A^{-1}x_{k-1}\|}$$

What does this converge to?

Why would anyone do that?

Inverse Iteration with Shift

What does this converge to?

$$x_k = \frac{(A - \sigma I)^{-1} x_{k-1}}{\|(A - \sigma I)^{-1} x_{k-1}\|}$$

Why would anyone do that?

Rayleigh Quotient Iteration

$$x_k = \frac{(A - \sigma I)^{-1} x_{k-1}}{\|(A - \sigma I)^{-1} x_{k-1}\|}$$