CS 357: Numerical Methods

Lecture 14:
Orthogonal Iteration
Singular Value Decomposition

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Finding Eigenvectors

X_k matrix → we would like columns to converge to eigenvectors

- Can we simultaneously (sort of) find all the eigenvectors of A?
- What about this algorithm

X_0 = arbitrary n x p matrix of rank p
for k=1,2,…
    $X_k = AX_{k-1}$

Problem: Overflow (could normalize)..

All columns converge to dominant eigenvector

Orthogonal Iteration

Q_k $n \times p$ orthogonal

R_k $p \times p$ upper Δ

- How about this?

X_0= n x p matrix of rank p
for k=1,2,….
    #compute reduced QR factorization
    $Q_{k+1}R_{k+1}=X_k$
    $X_{k+1}=AQ_{k+1}$

$$Q_1 R_1 = X_0$$
$$X_1 = A Q$$
$$Q_2 R_2 = X_1$$
$$X_2 = Q_2 A$$

Orthogonal Iteration

X_0= n x p matrix of rank p
for k=1,2,….
    #compute reduced QR factorization
    $Q_{k+1}R_{k+1}=X_k$
    $X_{k+1}=AQ_{k+1}$

Convergence $\underbrace{X_k \approx X_{k+1}}$

$$X_{k+1} = A\,Q_{k+1}$$

$$Q_{k+1} R_{k+1} = X_k$$

$$Q_{k+1} R_{k+1} \approx A\,Q_{k+1}$$

$$Q_{k+1} R_{k+1} Q^T_{k+1} \approx A$$

$$Q R Q^T = A$$

$$\boxed{Q R Q^{-1} = A}$$

Similarity Transform

R & A same eigenvalues

A is upper Δ

$\Rightarrow$ diagonal d_i are eigenvalues

The Schur Form: Finding Eigenvalues

$A = QRQ^T$

Eigenvalues of R = Eigenvalues of A

Demo

$EX = D$

$\underline{X = E^{-1} D}$

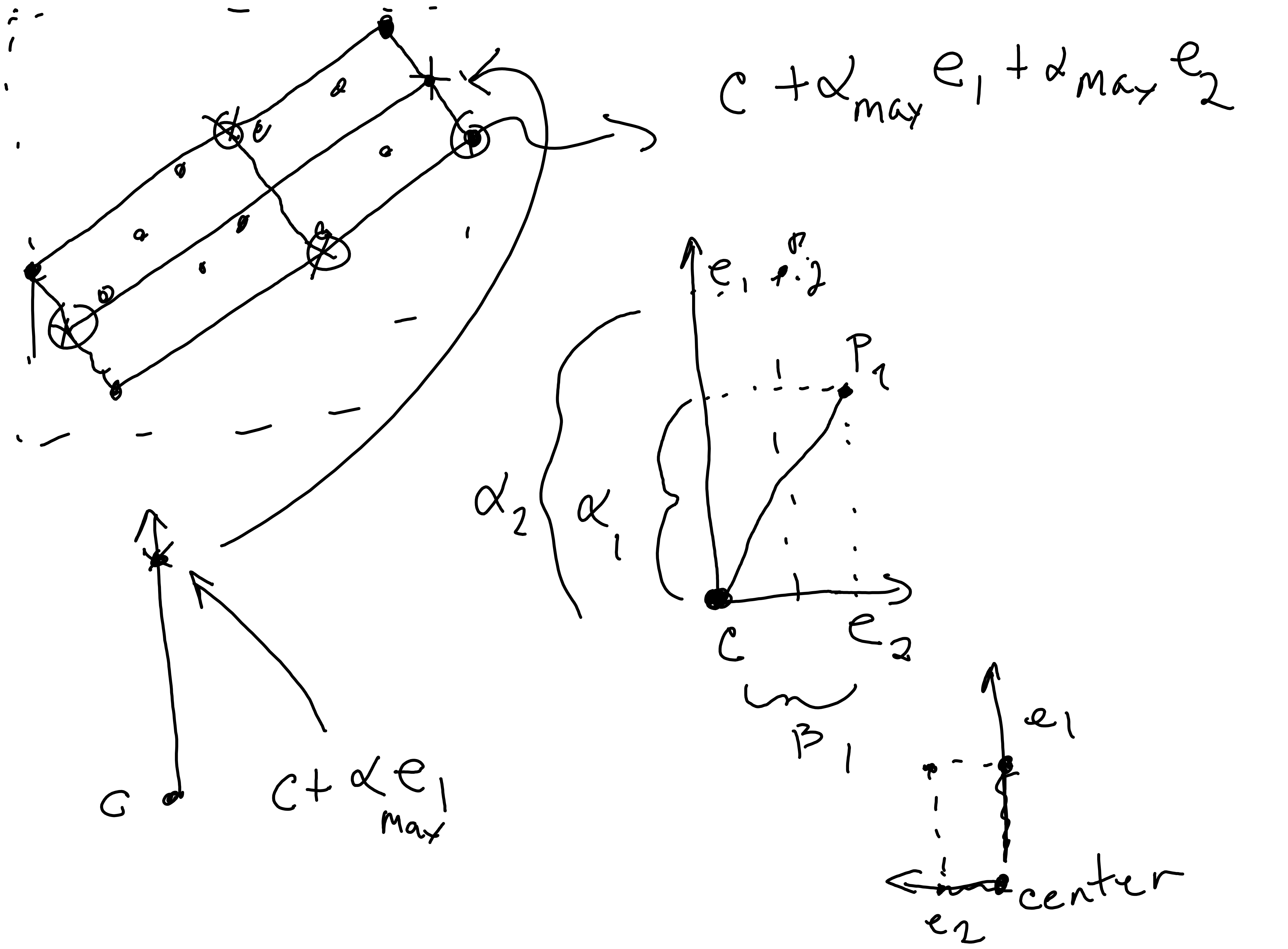
$A = XE$

$= E^{-1} D E$

Similarity Transform

D & A have same eigenvalues

Demo :


$c + \alpha_{max} e_1 + \alpha_{max} e_2$
e_1
p_2
P_1
α_2
α_1
c
e_2
β_1
c
$c + \alpha_{max} e_1$
e_1
center
e_2

The Schur Form: Finding Eigenvectors

$A = QRQ^T$

Eigenvectors of R : $\underbrace{Rx_i = \lambda x_i}$

$x_i \in N(R - \lambda I)$

we can show $\underbrace{y_i = Qx_i}$ eigenvector of A

$$Ay = AQx$$

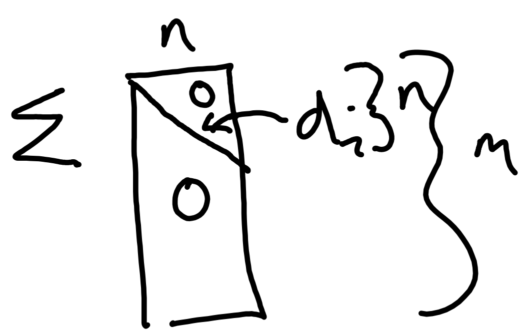
$$= QR\underbrace{Q^T Q}x = QRx$$

$$= Q\lambda x = \lambda \underbrace{Qx}_{y}$$

$$= \lambda y$$

Singular Value Decomposition (SVD)

A $m \times n$



$$A = U\Sigma V^T$$

V^T: $n \times n$ orthogonal

Σ: $m \times n$ sort of diagonal

U: $m \times m$ orthogonal

d_i are in descending order

Columns of V are right singular vectors

Columns of U are left " "

d_i singular values

Singular Value Decomposition (SVD)

$$A = U\Sigma V^T$$

Inversion using SVD

$$\underbrace{A = U\Sigma V^T}$$

$$\Sigma = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$\Sigma^{-1} = \begin{pmatrix} 1/d_1 & 0 \\ 0 & 1/d_2 \end{pmatrix}$$

Assume A is an n x n matrix

$$AA^{-1} = I$$

$$U\Sigma V^T A^{-1} = I$$

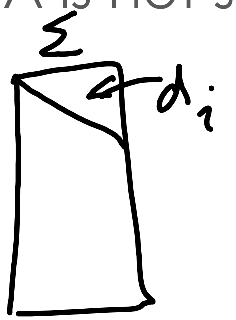
$$\Sigma V^T A^{-1} = U^T$$

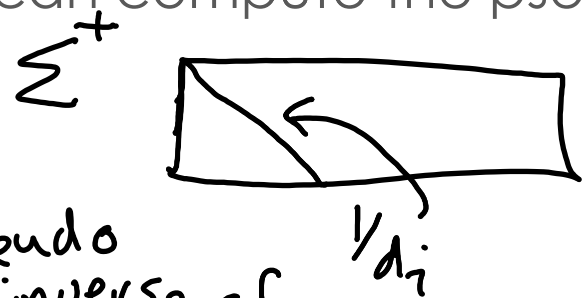
$$V^T A^{-1} = \Sigma^{-1} U^T$$

$$\boxed{A^{-1} = V\Sigma^{-1} U^T}$$

The Pseudo Inverse

- When A is not square we can compute the pseudo-inverse





pseudo inverse of Σ

$$A^+ = V\Sigma^+ U^T \Rightarrow \text{pseudo inverse of } A$$

Least Squares

$$Ax = b \qquad x = A^+ b$$

Pseudo Inverse and Least Squares

Notes on Computing the SVD