CS 357: Numerical Methods

Lecture 15: Singular Value Decomposition (SVD) SVD Applications

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But First…More Eigenvalues

Markov Chains

Describe a discrete system of states and transitions

(it’s a graph)

The Markov Property:
Only the current state matters in determining the probability of moving to another state

Markov Chains

- m_{ij} is the probability of moving from state ~~i~~ j to state ~~j~~ i

from state

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{ccc} \text{surf} & \text{study} & \text{eat} \\ \end{array}$$

$$A = \begin{pmatrix} .8 & .6 & .8 \\ .2 & .3 & 0 \\ 0 & .1 & .2 \end{pmatrix} \begin{array}{l} \text{surf} \\ \text{study} \\ \text{eat} \end{array}$$

to state

The columns add up to 1 because in each state the probabilities of the next states must add up to 1.

State Transitions

Modeled by a vector matrix product.

In each vector $<v_1, v_2, \ldots, v_n>$, v_i indicates the probability of being in state i

$$A \underset{\text{surf}}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} .8 \\ .2 \\ 0 \end{pmatrix} \qquad A \begin{pmatrix} .5 \\ .5 \\ 0 \end{pmatrix} = \begin{pmatrix} .4 + .3 \\ .1 + .15 \\ .05 \end{pmatrix}$$

Equilibrium

- Equilibrium is achieved when the probabilities do not change when we compute Av

  In other words $Av = \lambda v$

Computing the SVD

Compute the eigenvalues and eigenvectors of ~~ATA~~ $A^T A$

Use Orthogonal iteration

Construct V using the eigenvectors as column vectors

Construct Σ using square roots of the eigenvalues

Find U from $A = U \Sigma V^T$

Computing the SVD

$$A = U\Sigma V^T = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & \end{pmatrix}$$

Singular values σ_i are $\sqrt{\lambda_i}$ where λ_i is eigenvalue of A^TA

u_i are eigenvectors of AA^T

v_i A^TA

$$\boxed{O(mn^2 + n^3 \text{ work for } m \times n\ A}$$

The Outer Product

$n \times 1$ $1 \times m$

What is uv^T ? Outer product → $n \times m$ matrix

Mathematically, vectors are thought of as nx1

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Another way of expressing $A=U\,\Sigma\,V^T$

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

$=$ sum of outer products of singular vectors

What do the Singular Values Mean?

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=1} \|\Sigma U^T x\|_2$$

$$\boxed{= \sigma_1}$$

Computing the Condition Number

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n$$

Rank-k Approximations to A

Rank k appoximation =
sum of first k outer products

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

A_k has rank k

Rank-k Approximations to A

$\|A - B\|_2$ minimized by $B = A_k$

$\|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$

A_k is best rank k approx. to A

The Froebinius Norm*

* Not really a matrix norm

$$\|A\|_F = \sqrt{a_{11}^2 + a_{12}^2 + \dots a_{1n}^2 + a_{21}^2 + \dots a_{mn}^2}$$

Treat A like a big vector and take sqrt of sum of squares

$$\min_{\operatorname{rank} B \leq k} \|A - B\|_F = \|A - A_k\|_F$$

$$= \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_n^2}$$