CS 357: Numerical Methods

Lecture 15: Singular Value Decomposition (SVD) SVD Applications

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But First...More Eigenvalues

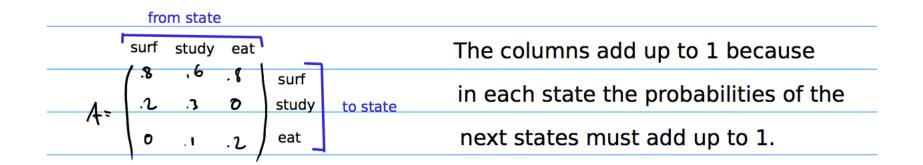
Markov Chains

Describe a discrete system of states and transitions (it's a graph)

The Markov Property: Only the current state matters in determining the probability of moving to another state

Markov Chains

\square m_{ij} is the probability of moving from state j to state $\frac{1}{3}$



State Transitions

Modeled by a vector matrix product.

In each vector $\langle v_1, v_2, ..., v_n \rangle$, v_i indicates the probability of being in state i

Equilibrium

Equilibrium is achieved when the probabilities do not change when we compute Av

In other words $Av = \lambda v$

Computing the SVD

Compute the eigenvalues and eigenvectors of A A Use Orthogonal iteration

Construct V using the eigenvectors as column vectors

Construct Σ using square roots of the eigenvalues

Find U from A=U ΣV^T

Computing the SVD

 $A = \mathcal{M} \not \geq \mathcal{V} = \begin{pmatrix} 1 & 1 & 1 \\ u_1 & u_2 & \dots & u_m \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_n \end{pmatrix} \begin{pmatrix} -\nu_1 - \nu_2 \\ -\nu_2 - \nu_2 \\ \vdots \\ -\nu_n \end{pmatrix}$ Singular values of are they where The is eigenvalue of ATA Un are eigenectors of AAT TOCMPTH work for MXNA

The Outer Product

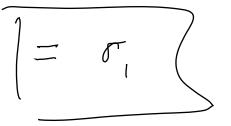
What is
$$uvi$$
? Outer product $\rightarrow A \times m$ matrix
Mathematically, vectors are thought of as $nx1$
 \mathcal{E}_{xample}
 $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ $=$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Another way of expressing $A=U \Sigma V^T$

 $A = \sum_{j=1}^{n} \sigma_j u_j v_j^{\top}$ = 5 mm of outer products of singular rectors

What do the Singular Values Mean?

$$||A_{1}|_{2} = \max_{\substack{||X_{1}|_{2}=1}} ||A_{1}||_{2} = \max_{\substack{||X_{1}|_{2}=1}} ||A_{1}||_{2} = \max_{\substack{||X_{1}|_{2}=1}} ||E_{1}||_{2}$$



Computing the Condition Number

$$k(A) = IIAII_2 IIAII_2 = \sigma_n$$

Rank-k Approximations to A

Rank & appoximation = Sum of first kouter products $A = \sum_{i=1}^{k} \sigma_i u_i v_i$ $K = \sum_{i=1}^{k} \sigma_i u_i v_i$ A has rank k

Rank-k Approximations to A

The Froebinius Norm*

* Not really a matrix norm ||All F = Jaz + az + az + az + ... am Treat A like a big vector and take sgit of sum of squares

 $m.n = \frac{\|A-B\|}{F} = \frac{\|A-A_k\|}{F}$ $= \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_n^2}$