CS 357: Numerical Methods

Lecture 1: Vectors and Python

Eric Shaffer

Adapted from the slides of Phillip Klein
Homework 0 Stuff

- Vector spaces
- Combinations
- Python
Much of linear algebra based just on +, -, *, / and algebraic properties

- / is inverse of *
- - is inverse of +
- addition is commutative: \(a + b = b + a\)
- multiplication distributes over addition: \(a \cdot (b + c) = a \cdot b + a \cdot c\)
- etc.

Such a collection of ”numbers” with +, -, *, / is called a field. Different fields are like different classes obeying the same interface.
A Field is a set (with operators) with certain properties

In the book, they discuss three fields:

... The field $\mathbb{R}$ of real numbers

... The field $\mathbb{C}$ of complex numbers

... The finite field $GF(2)$, which consists of 0 and 1 under mod 2 arithmetic.
Playing with $GF(2)$

Galois Field 2 has just two elements: 0 and 1

Addition is like exclusive-or:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Multiplication is like ordinary multiplication:

<table>
<thead>
<tr>
<th>$\times$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Usual algebraic laws still hold, e.g. multiplication distributes over addition

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
What is a vector space?

A set of objects defined over a Field

- Usually the field we will work will be the real numbers

Two operations: vector addition and scalar multiplication

To show a set $S$ is a vector space you need to:

- Show that there is an additive identity element in $S$
- The set is closed under vector addition
- The set is closed under scalar multiplication
Example

- What about set of vectors \((a, a/2)\)?
- \((0,0)\) is in the set and \((0,0)+(a,a/2) = (a,a/2)\)
  - Additive Identity
- \((a,a/2)+(b,b/2) = (a+b, (a+b)/2)\) is in the set
  - Closed under addition
- \(\alpha (a,a/2) = (\alpha a, \alpha a/2)\) is in the set
  - Closed under multiplication
Example

- What about \((a,b)\) such that \(a+b=1\) ?
- What about \((a,b)\) such that \(a+b=0\) ?
Combinations

- Combinations are weighted sums of vectors.
- Constraints on the weights determine the type of combination.
  - Linear
    - Weights are real numbers.
  - Affine
    - Weights are reals that sum to 1.
  - Convex
    - Weights are reals that are non-negative and sum to 1.
Linear Combinations

An expression

\[ \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \]

is a *linear combination* of the vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \).

The scalars \( \alpha_1, \ldots, \alpha_n \) are the *coefficients* of the linear combination.

**Example:** One linear combination of \([2, 3.5]\) and \([4, 10]\) is

\[-5 [2, 3.5] + 2 [4, 10]\]

which is equal to \([-5 \cdot 2, -5 \cdot 3.5] + [2 \cdot 4, 2 \cdot 10]\]

Another linear combination of the same vectors is

\[0 [2, 3.5] + 0 [4, 10]\]

which is equal to the zero vector \([0, 0]\).

**Definition:** A linear combination is *trivial* if the coefficients are all zero.
Definition: The set of all linear combinations of some vectors $v_1, \ldots, v_n$ is called the span of these vectors.

Written Span $\left\{v_1, \ldots, v_n\right\}$. 
Generators

**Definition:** Let $V$ be a set of vectors. If $v_1, \ldots, v_n$ are vectors such that $V = \text{Span}\{v_1, \ldots, v_n\}$ then

1. we say $\{v_1, \ldots, v_n\}$ is a generating set for $V$;
2. we refer to the vectors $v_1, \ldots, v_n$ as generators for $V$.

**Example:** $\{[3, 0, 0], [0, 2, 0], [0, 0, 1]\}$ is a generating set for $\mathbb{R}^3$.

**Proof:** Must show two things:
1. Every linear combination is a vector in $\mathbb{R}^3$.
2. Every vector in $\mathbb{R}^3$ is a linear combination.

First statement is easy: every linear combination of 3-vectors over $\mathbb{R}$ is a 3-vector over $\mathbb{R}$, and $\mathbb{R}^3$ contains all 3-vectors over $\mathbb{R}$.

Proof of second statement: Let $[x, y, z]$ be any vector in $\mathbb{R}^3$. I must show it is a linear combination of my three vectors....

$$[x, y, z] = \left(\frac{x}{3}\right)[3, 0, 0] + \left(\frac{y}{2}\right)[0, 2, 0] + z[0, 0, 1]$$
**Definition:** If \( \mathbf{c} \) is a vector and \( V \) is a vector space then
\[
\mathbf{c} + V
\]
is called an *affine space*.

**Examples:** A plane or a line not necessarily containing the origin.
A linear combination $\gamma \mathbf{u}_1 + \alpha \mathbf{u}_2 + \beta \mathbf{u}_3$ where $\gamma + \alpha + \beta = 1$ is an affine combination.

The set of all affine combinations of a set of vectors is the affine hull

- Affine hull of two vectors is a line
- Affine hull of three vectors is a plane
- Affine hull of one vector is what?

Notice anything about the dimension of the object produced?
Geometry of sets of vectors: convex hull

**Earlier, we saw:** The \( \mathbf{u} \)-to-\( \mathbf{v} \) line segment is

\[
\{ \alpha \mathbf{u} + \beta \mathbf{v} : \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1 \}
\]

**Definition:** For vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) over \( \mathbb{R} \), a linear combination

\[
\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n
\]

is a *convex combination* if the coefficients are all nonnegative and they sum to 1.

... Convex hull of a single vector is a point.
... Convex hull of two vectors is a line segment.
... Convex hull of three vectors is a triangle

Convex hull of more vectors is what?