CS 357: Numerical Methods

Lecture 1: Vectors and Python

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Adapted from the slides of Phillip Klein

Homework 0 Stuff

- Vector spaces
- Combinations
- Python

Three slides of Abstract Algebra

Much of linear algebra based just on +, -, *, / and algebraic properties

- / is inverse of *
- \- is inverse of +
- *addition is commutative:* $a + b = b + a$
- *multiplication distributes over addition:* $a * (b + c) = a * b + a * c$
- etc.

Such a collection of "numbers" with +, -, *, / is called a *field*. Different fields are like different classes obeying the same interface.

A Field is a set (with operators) with certain properties

In the book, they discuss three fields:

- The field R of real numbers
- The field C of complex numbers
- The finite field $GF(2)$, which consists of 0 and 1 under mod 2 arithmetic.

Playing with $GF(2)$

Galois Field 2
has just two elements: 0 and 1

Addition is like exclusive-or:

+	0	1
0	0	1
1	1	0

Multiplication is like ordinary multiplication

×	0	1
0	0	0
1	0	1

Evariste Galois, 1811-1832

Usual algebraic laws still hold, e.g. multiplication distributes over addition

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Vector Spaces

- What is a vector space?
- A set of objects defined over a Field
  - Usually the field we will work will be the real numbers
- Two operations: vector addition and scalar multiplication
- To show a set S is a vector space you need to:
  - Show that there is an additive identity element in S
  - The set is closed under vector addition
  - The set is closed under scalar multiplication

Example

- What about set of vectors (a, a/2)?
- (0,0) is in the set and (0,0)+(a,a/2) = (a,a/2)
  - Additive Identity
- (a,a/2)+(b,b/2) = ((a+b), (a+b)/2) is in the set
  - Closed under addition
- α (a,a/2) = (α a, α a/2) is in the set
  - Closed under multiplication

Example

- What about (a,b) such that a+b=1 ?
- What about (a,b) such that a+b=0 ?

Combinations

- *Combinations* are weighted sums of vectors
- Constraints on the weights determine the type of combination
- Linear
  - Weights are real numbers
- Affine
  - Weights are reals that sum to 1
- Convex
  - Weights are reals that are non-negative and sum to 1

Linear Combinations

An expression

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

is a *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

The scalars $\alpha_1, \ldots, \alpha_n$ are the *coefficients* of the linear combination.

Example: One linear combination of $[2, 3.5]$ and $[4, 10]$ is

$$-5\,[2, 3.5] + 2\,[4, 10]$$

which is equal to $[-5 \cdot 2, -5 \cdot 3.5] + [2 \cdot 4, 2 \cdot 10]$

Another linear combination of the same vectors is

$$0\,[2, 3.5] + 0\,[4, 10]$$

which is equal to the zero vector $[0, 0]$.

Definition: A linear combination is *trivial* if the coefficients are all zero.

Span

Definition: The set of all linear combinations of some vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called the
span of these vectors

Written Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Generators

Definition: Let V be a set of vectors. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are vectors such that $V = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then

- we say $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a *generating set* for V;
- we refer to the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as *generators* for V.

Example: $\{[3, 0, 0], [0, 2, 0], [0, 0, 1]\}$ is a generating set for $\mathbb{R}^3$.

Proof: Must show two things:

1. Every linear combination is a vector in $\mathbb{R}^3$.
2. Every vector in $\mathbb{R}^3$ is a linear combination.

First statement is easy: every linear combination of 3-vectors over R is a 3-vector over
R, and $\mathbb{R}^3$ contains all 3-vectors over R.

Proof of second statement: Let $[x, y, z]$ be any vector in $\mathbb{R}^3$. I must show it is a linear combination of my three vectors....

$$[x, y, z] = (x/3)\,[3, 0, 0] + (y/2)\,[0, 2, 0] + z\,[0, 0, 1]$$

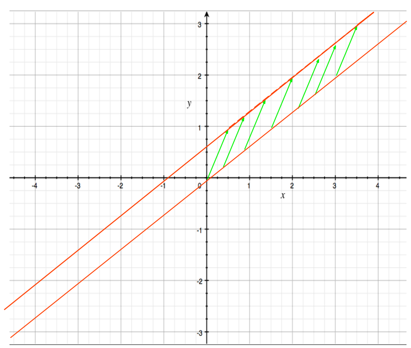
Affine space

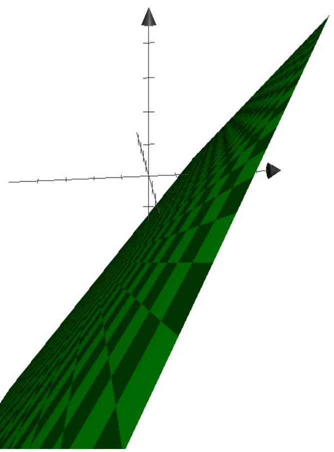
Definition: If $\mathbf{c}$ is a vector and V is a vector space then

$$\mathbf{c} + V$$

is called an *affine space*.

Examples: A plane or a line not necessarily containing the origin.





Affine Combinations

- A linear combination $\gamma\, \mathbf{u}_1 + \alpha\, \mathbf{u}_2 + \beta\, \mathbf{u}_3$ where $\gamma + \alpha + \beta = 1$ is an *affine combination*.

- The set of all affine combinations of a set of vectors is the *affine hull*
  - Affine hull of two vectors is a line
  - Affine hull of three vectors is a plane
  - Affine hull of one vector is what?

- Notice anything about the dimension of the object produced?

Geometry of sets of vectors: convex hull

Earlier, we saw: The **u**-to-**v** line segment is

$$\{\alpha\,\mathbf{u} + \beta\,\mathbf{v} \ : \ \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \alpha \geq 0, \beta \geq 0, \alpha + \beta = 1\}$$

Definition: For vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ over R, a linear combination

$$\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$$

is a *convex combination* if the coefficients are all nonnegative and they sum to 1.

- Convex hull of a single vector is a point.
- Convex hull of two vectors is a line segment.
- Convex hull of three vectors is a triangle

Convex hull of more vectors is what?