CS 357: Numerical Methods

Lecture 3: Matrices and Vector Norms

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Adapted from the slides of Phillip Klein

Matrices

What is a matrix? Traditional answer

Neo: What is the Matrix?
Trinity: The answer is out there, Neo, and it's looking for you, and it will find you if you want it to. *The Matrix*, 1999

Traditional notion of a matrix: two-dimensional array.

$$\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$$

- Two rows: $[1, 2, 3]$ and $[10, 20, 30]$.
- Three columns: $[1, 10]$, $[2, 20]$, and $[3, 30]$.
- A 2×3 matrix.

For a matrix A, the i, j element of A

- is the element in row i, column j
- is traditionally written $A_{i,j}$
- but we will use $A[i, j]$

Column space and row space

One simple role for a matrix: packing together a bunch of columns or rows

Two vector spaces associated with a matrix M:
Definition:

- *column space* of M = Span {columns of M}
  Written Col M

- *row space* of M = Span {rows of M}
  Written Row M

Examples:

- Column space of $\begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix}$ is Span $\{[1, 10], [2, 20], [3, 30]\}$.
  In this case, the span is equal to Span $\{[1, 10]\}$ since $[2, 20]$ and $[3, 30]$ are scalar multiples of $[1, 10]$.

- The row space of the same matrix is Span $\{[1, 2, 3], [10, 20, 30]\}$.
  In this case, the span is equal to Span $\{[1, 2, 3]\}$ since $[10, 20, 30]$ is a scalar multiple of $[1, 2, 3]$.

Transpose

Transpose swaps rows and columns.

```
          @   #   ?
      ---------
a   |   2   1   3
b   |  20  10  30
```

→

```
          a   b
      ------
@   |    2  20
#   |    1  10
?   |    3  30
```

Matrices are Vectors

Matrices as vectors

Soon we study true matrix operations. But first....

A matrix can be interpreted as a vector:

- an $R \times S$ matrix is a function from $R \times S$ to $\mathbb{F}$,
- so it can be interpreted as an $R \times S$-vector:
  - *scalar-vector multiplication*
  - *vector addition*

Null Space

Null space of a matrix

Definition: *Null space* of a matrix A is $\{\mathbf{u} \; : \; A * \mathbf{u} = \mathbf{0}\}$. Written *Null A*

Example:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * [0,0,0] = [0,0]$$

so the null space includes $[0,0,0]$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * [6,-1,-1] = [0,0]$$

so the null space includes $[6,-1,-1]$

By dot-product definition,

$$\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix} * \mathbf{u} = [\mathbf{a}_1 \cdot \mathbf{u}, \; \ldots, \; \mathbf{a}_m \cdot \mathbf{u}]$$

Null space of a matrix

We just saw:

Null space of a matrix $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

equals the solution set of the homogeneous linear system $\boxed{\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}}$

This shows: *Null space of a matrix is a vector space.*

Can also show it directly, using algebraic properties of matrix-vector multiplication:

Property V1: Since $A * \mathbf{0} = \mathbf{0}$,the null space of A contains $\mathbf{0}$

Property V2: if $\mathbf{u} \in \text{Null } A$ then $A * (\alpha \, \mathbf{u}) = \alpha \, (A * \mathbf{u}) = \alpha \, \mathbf{0} = \mathbf{0}$ so $\alpha \mathbf{u} \in \text{Null } A$

Property V3: If $\mathbf{u} \in \text{Null } A$ and $\mathbf{v} \in \text{Null } A$
then $A * (\mathbf{u} + \mathbf{v}) = A * \mathbf{u} + A * \mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$
so $\mathbf{u} + \mathbf{v} \in \text{Null } A$

Solution space of a matrix-vector equation

Earlier, we saw:

If $\mathbf{u}_1$ is a solution to the linear system $\boxed{\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & \beta_m \end{array}}$

then the solution set is $\mathbf{u}_1 + \mathcal{V}$,

where $\mathcal{V}$ = solution set of $\boxed{\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}}$

Restated: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$
where $\mathcal{V} = \text{Null } A$

Solution space of a matrix-vector equation

Proposition: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V} = \text{Null } A$

Example:

- Null space of $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix}$ is Span $\{[6, -1, -1]\}$.
- One solution to $\boxed{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * \mathbf{x} = [1, 1]}$ is $x = [-1, 1, 0]$.
- Therefore solution set is $[-1, 1, 0] +$ Span $\{[6, -1, -1]\}$
- For example, solutions include
  - $[-1, 1, 0] + [0, 0, 0]$
  - $[-1, 1, 0] + [6, -1, -1]$
  - $[-1, 1, 0] + 2\,[6, -1, -1]$

$$\vdots$$

Solution space of a matrix-vector equation

Proposition: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V} = \text{Null } A$

- If $\mathcal{V}$ is a trivial vector space then $\mathbf{u}_1$ is the only solution.
- If $\mathcal{V}$ is not trivial then $\mathbf{u}_1$ is *not* the only solution.

Corollary: $A * \mathbf{x} = \mathbf{b}$ has at most one solution iff Null A is a trivial vector space.

Question: How can we tell if the null space of a matrix is trivial?

Answer comes later...

Gaussian Elimination Review
Solving Systems by Hand

The generic lower and upper triangular matrices are

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ l_{n1} & & \cdots & l_{nn} \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & u_{nn} \end{bmatrix}$$

The triangular systems

$$Ly = b \qquad Ux = c$$

are easily solved by **forward substitution** and **backward substitution**, respectively

Non-Triangular Example

Solve

$$x_1 + 3x_2 = 5$$
$$2x_1 + 4x_2 = 6$$

Subtract 2 times the first equation from the second equation

$$x_1 + 3x_2 = 5$$
$$-2x_2 = -4$$

This equation is now in triangular form, and can be solved by backward substitution.

Non-Triangular Example

The elimination phase transforms the matrix and right hand side to an equivalent system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 + 4x_2 &= 6 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x_1 + 3x_2 &= 5 \\ -2x_2 &= -4 \end{aligned}$$

The two systems have the same solution. The right hand system is upper triangular.
Solve the second equation for x_2

$$x_2 = \frac{-4}{-2} = 2$$

Substitute the newly found value of x_2 into the first equation and solve for x_1.

$$x_1 = 5 - (3)(2) = -1$$

3 by 3 Example

When performing Gaussian Elimination by hand, we can avoid copying the x_i by using a shorthand notation.
For example, to solve:

$$A = \begin{bmatrix} -3 & 2 & -1 \\ 6 & -6 & 7 \\ 3 & -4 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ -7 \\ -6 \end{bmatrix}$$

Form the *augmented* system

$$\tilde{A} = [A \ \ b] = \left[\begin{array}{rrr|r} -3 & 2 & -1 & -1 \\ 6 & -6 & 7 & -7 \\ 3 & -4 & 4 & -6 \end{array}\right]$$

The vertical bar inside the augmented matrix is just a reminder that the last column is the b vector.

3 by 3 Example

Add 2 times row 1 to row 2, and add (1 times) row 1 to row 3

$$\tilde{A}_{(1)} = \left[\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & -2 & 3 & -7 \end{array}\right]$$

Subtract (1 times) row 2 from row 3

$$\tilde{A}_{(2)} = \left[\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

3 by 3 Example

The transformed system is now in upper triangular form

$$\tilde{A}_{(2)} = \left[\begin{array}{ccc|c} -3 & 2 & -1 & -1 \\ 0 & -2 & 5 & -9 \\ 0 & 0 & -2 & 2 \end{array}\right]$$

Solve by back substitution to get

$$x_3 = \frac{2}{-2} = -1$$

$$x_2 = \frac{1}{-2}(-9 - 5x_3) = 2$$

$$x_1 = \frac{1}{-3}(-1 - 2x_2 + x_3) = 2$$

Finding an Inverse via Gaussian Elimination

- Append the identity matrix to the given matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$$[A|I] = \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Finding an Inverse via Gaussian Elimination

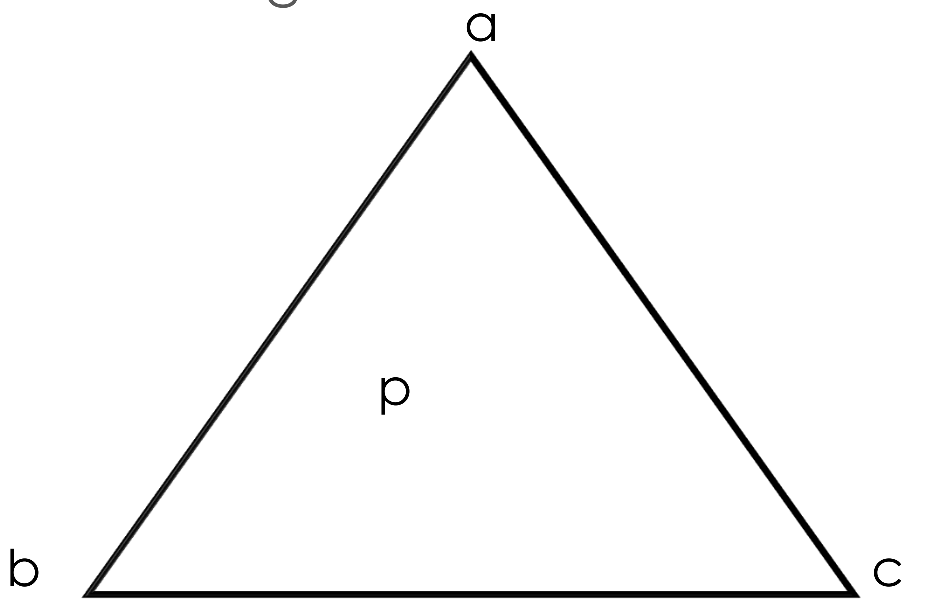
- Transform the Left Hand Side (LHS) to transform it to the identity matrix

$$[A|I] = \left[\begin{array}{rrr|rrr} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right].$$

$$[I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right].$$

Barycentric Coordinates for Triangles

- Describe location of point in a triangle in relation to the vertices
- $p=(\lambda_1, \lambda_2, \lambda_3)$ where the following are true
  - $p=\lambda_1 a + \lambda_2 b + \lambda_3 c$
  - $\lambda_1+\lambda_2+\lambda_3= 1$



Digression: Barycentric Coordinates

- Grab some code….

  http://courses.engr.illinois.edu/cs519/CS357/barycentric.py

- Questions:
  - What kind of combination are barycentric coordinates
  - What kind of combination are the “inside” points
  - Any ideas about what applications barycentric coordinates might have?

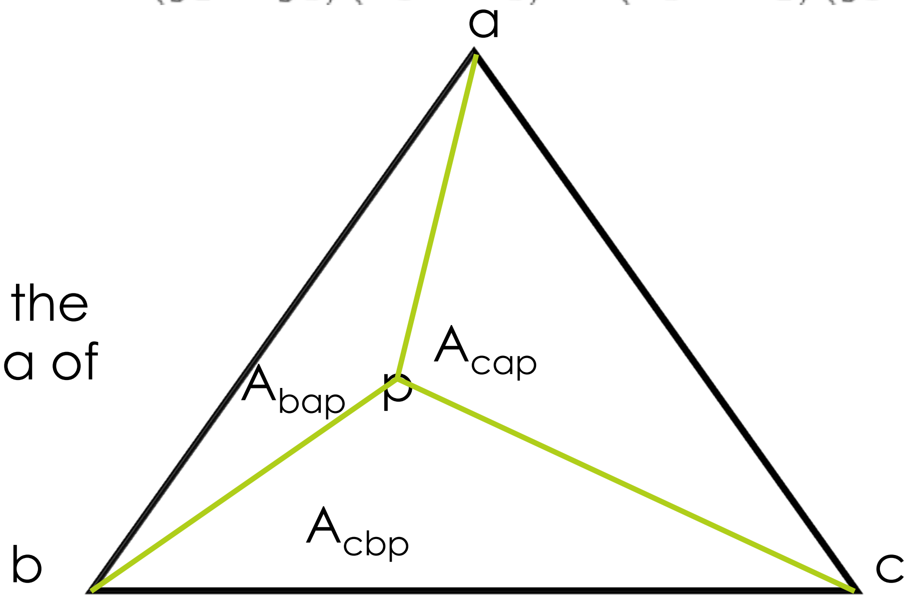
Barycentric Coordinates for Triangles

$$\lambda_1 = \frac{(y_2 - y_3)(x - x_3) + (x_3 - x_2)(y - y_3)}{\det(T)} = \frac{(y_2 - y_3)(x - x_3) + (x_3 - x_2)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)},$$

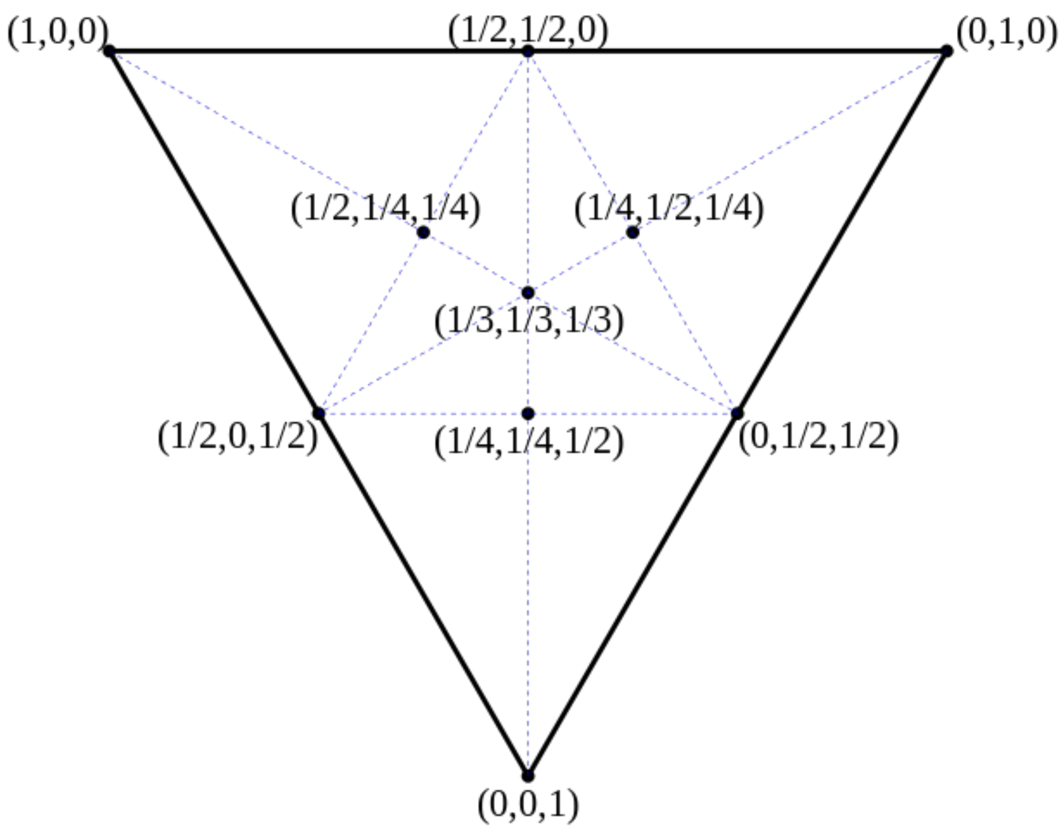
$$\lambda_2 = \frac{(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)}{\det(T)} = \frac{(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)},$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2 \,.$$

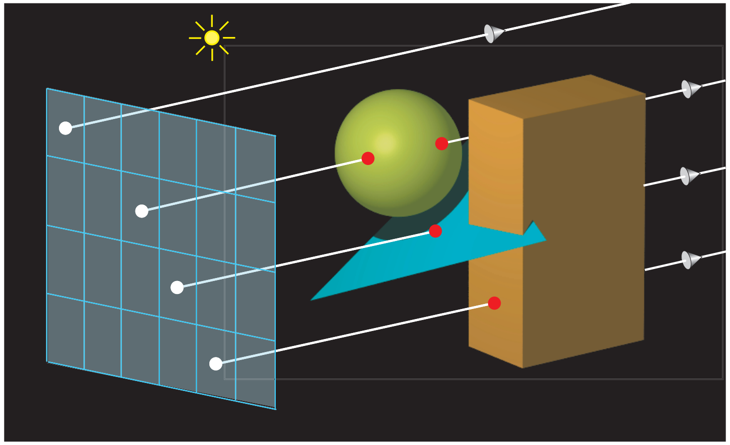
Coordinates are the signed area of the opposite subtriangle divided by area of the triangle

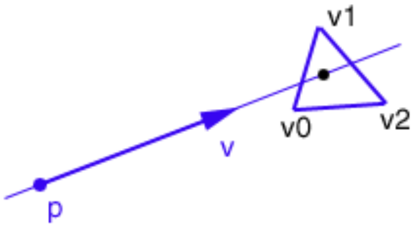


Barycentric Coordinates for Triangles

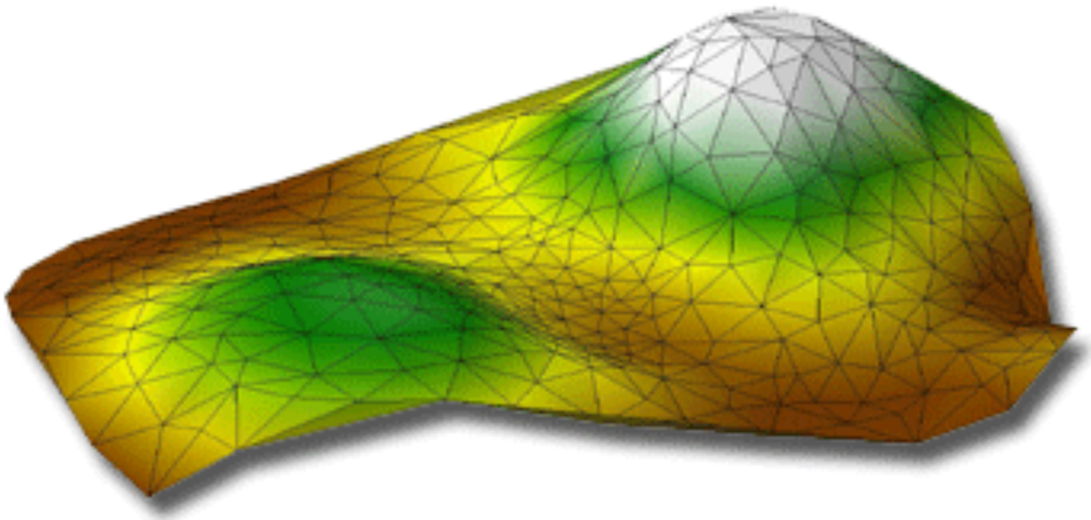


Point in Triangle Test for Production Rendering





Interpolating Functions on Triangles



- To interpolate a function sampled at the vertices we just do:

$$\mathbf{f(p)} = \lambda_1 \mathbf{f(a)} + \lambda_2 \mathbf{f(b)} + \lambda_3 \mathbf{f(c)}$$

inside the triangle…..

Vector Norms…

Vector norms are functions that map a vector to a real number

You can think of it as measuring the magnitude of the vector

The norm you know is the 2-norm:

You can use it to measure the distance between two points

Vector Norms…

Vector norms are functions that map a vector to a real number

You can think of it as measuring the magnitude of the vector

The norm you know is the 2-norm:

$$\|v\|_2 = \sqrt{\sum v_i^2}$$

$$v = \langle v_0, v_1, \ldots, v_{d-1} \rangle$$

You can use it to measure the distance between two points

Compute a vector v = p2-p1 and take the norm of v

Vector Norms…

Vector norms are functions that map a vector to a real number

You can think of it as measuring the magnitude of the vector

Vectors:

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p}$$

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max\left(|x_1|, |x_2|, \ldots, |x_n|\right) = \max_i\left(|x_i|\right)$$

Vector Norms…some properties

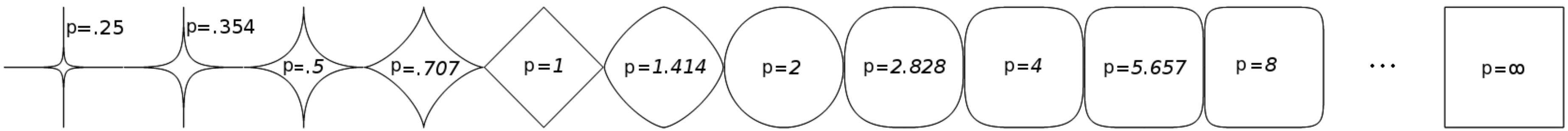
$$\|\alpha x\| = |\alpha| \|x\|$$

$$\|Ax\| \leqslant \|A\| \|x\|$$

Triangle Inequality $\|x + y\| \leqslant \|x\| + \|y\|$

Visualizing 1-ball of Norms

Where does the norm equal 1 in the 2D Euclidean plane



Matrix Norms

Matrices:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

$$\|A\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leqslant i \leqslant m} \sum_{j=1}^{n} |a_{ij}|$$