CS 357: Numerical Methods

Lecture 4: Matrix Norms

Eric Shaffer

Vector Norms…

Vector norms are functions that map a vector to a real number

You can think of it as measuring the magnitude of the vector

The norm you know is the 2-norm:

$$\|v\|_2 = \sqrt{\sum v_i^2}$$

$$v = \langle v_0, v_1, \ldots, v_{d-1} \rangle$$

You can use it to measure the distance between two points

Compute a vector v = p2-p1 and take the norm of v

Requirements for a Norms…

All of the following must be true for some function to be a norm

$$\|x\| \geq 0$$

$$\|x + y\| \leq \|x\| + \|y\|$$

$$\|ax\| = |a| \|x\|$$

$$\|x\| = 0 \Leftrightarrow x = 0$$

Vector Norms…

Vector norms are functions that map a vector to a real number

You can think of it as measuring the magnitude of the vector

Vectors:

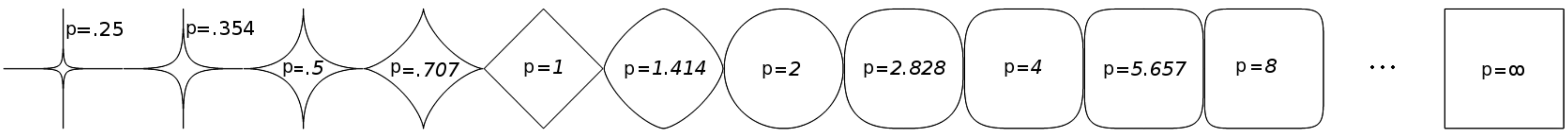
$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p}$$

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n| = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max\left(|x_1|, |x_2|, \ldots, |x_n|\right) = \max_i\left(|x_i|\right)$$

Visualizing 1-ball of Norms

Where does the norm equal 1 in the 2D Euclidean plane



Measuring Error

- Norms measure distance between vectors
  - The distance between v1 and v2 is $\|v1 - v2\|$
- Imagine we have an approximation x_0 to some true value x
  So…$x = x_0 + \Delta x$
- The absolute error is Δx
- …for vectors we could use $\|\Delta x\|$
- Can this tell us if an answer is good?

Absolute Error - Example

$$|x - x_0| = 0.1$$

if $x = 1000\ 000$

if $x = 0.1$ ~ bad

Relative Error

- Relative error is defined as the ***absolute error divide by the true value***

$$\boldsymbol{error_{rel}} = \frac{||\Delta \boldsymbol{x}||}{||\boldsymbol{x}||}$$

- This allows us some way to assess the magnitude or error
- Any obvious issues with computing error?

Relative Error

- We usually don't know the true value x
- So….
  - If we know an error range, compute an upper bound
  - Take the largest possible error value use that

Precision and Accuracy

- **Precision:** number of digits with which a number is expressed
- **Accuracy:**
  number of correct significant digits in approximating a value

3.141 Δ = 0.001 ish
$\simeq 10^{-3}$

- What about saying π = 3.1425289898?
- Rule of thumb, if a number has a relative error of about 10^{-p} then it has about p correct significant digits in its decimal representation

Relative Error - Example

$1/8 = 0.125 \approx .12$

$\Delta x = 0.005$

$\text{rel} = \frac{0.005}{0.125} = 0.04$

4% error

Condition Number

- We'll be interested in numerical methods (algorithms)
- How much does a method amplify the error in an input
  - Every input has some error
  - Every output has some error…
- **Condition number** is the maximum ratio of output error to input error over all possible inputs

$$C = \max_{x_{in}} \frac{\|\Delta x_{out}\|}{\|\Delta x_{in}\|}$$

Matrices Transformations (3D Graphics version)

- General

$$\begin{bmatrix} d & e & f & a \\ g & h & i & b \\ j & k & l & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} dx + ey + fz + a \\ gx + hy + iz + b \\ jx + ky + lz + c \\ 1 \end{bmatrix}$$

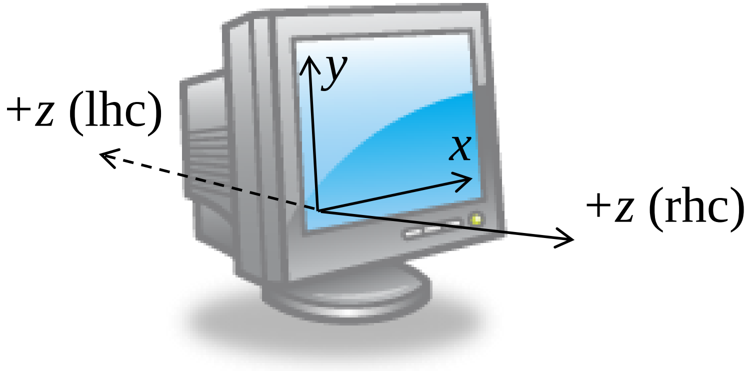
- Translation

$$\begin{bmatrix} 1 & & & a \\ & 1 & & b \\ & & 1 & c \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + a \\ y + b \\ z + c \\ 1 \end{bmatrix}$$

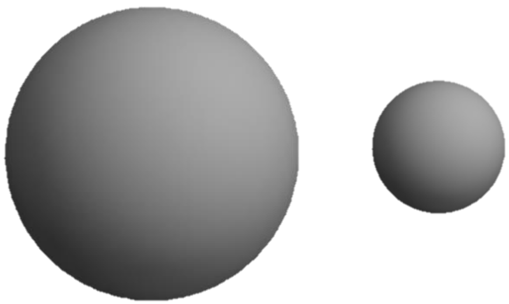
3-D Coordinates

- Points represented by 4-vectors
- Need to decide orientation of coordinate axes

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

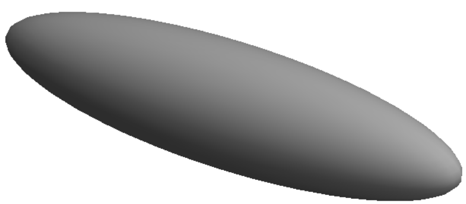


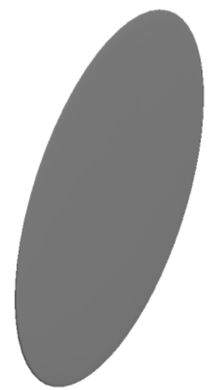
Scale



Uniform Scale
$a = b = c = ¼$

$$\begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$$





Stretch
$a = b = 1$, $c = 4$

Squash
$a = b = 1$, $c = ¼$

Project
$a = b = 1$, $c = 0$

Invert
$a = b = 1$, $c = -1$

3-D Rotations

- About x-axis
  - rotates $y \rightarrow z$

$$\begin{bmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix}$$

- About y-axis
  - rotates $z \rightarrow x$

$$\begin{bmatrix} \cos\theta & & \sin\theta & \\ & 1 & & \\ -\sin\theta & & \cos\theta & \\ & & & 1 \end{bmatrix}$$

- About z-axis
  - rotates $x \rightarrow y$

$$\begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

- Rotations do not commute!

Norms of Linear Functions

- What is a linear function?

$$f(x_1, \ldots, x_k) = b + a_1 x_1 + \ldots + a_k x_k$$

- …a hyperplane in dimension k
- We can define the norm of a linear function as

$$\|f\| = \max \frac{\|f(x)\|}{\|x\|}$$

With x ≠0 (Why?)

- This is like a maximum amplification factor

Matrix Norm

- We'll define a matrix norm to be the max possible stretching a matrix can perform on a vector….

$$||A|| = \max||Ax|| \text{ for all } x \text{ with } ||x|| = 1$$

Matrix Norms

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

$$\|A\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leqslant i \leqslant m} \sum_{j=1}^{n} |a_{ij}|$$

Matrix norms are defined in terms of an underlying vector norm

In general matrix p-norms are complicated to compute but a couple of them are easy…

What is the 1-norm computing?

What is the inf-norm computing?

Matrix Norms - Example

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$

$$\|A\|_1 = 6$$

$$\|A\|_\infty = 8$$

Solving Linear Systems and Error

- We want solve Ax=b for x
- How much does the algorithm amplify error?

$$A(x + \Delta x) = (b + \Delta b)$$

$A\Delta x = \Delta b \rightarrow$

$\Delta x = A^{-1}\Delta b$

Δ input error

output error

$$\frac{\|\Delta x\|}{\|x\|} \Big/ \frac{\|\Delta b\|}{\|b\|} \leq \|A\|\|A^{-1}\|$$

condition number

Solving Linear Systems and Error

$$\frac{\|\Delta x\|}{\|x\|} \Big/ \frac{\|\Delta b\|}{\|b\|} \leq \|A\| \|A^{-1}\|$$

$$\frac{\Delta x}{x} \Big/ \frac{\Delta b}{b} = \frac{\|\Delta x\| \; \|b\|}{\|x\| \; \|\Delta b\|} = \frac{\|b\|}{\|x\|} \frac{\|\Delta x\|}{\|\Delta b\|}$$

$=$ since $Ax = b$

$$\frac{\|Ax\|}{\|x\|} \quad \frac{\|\Delta x\|}{\|\Delta b\|}$$

$$= \qquad \frac{\|A^{-1}\Delta b\|}{\|\Delta b\|}$$