CS 357: Numerical Methods

Lecture 5: Condition Number
Gaussian Elimination Review

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Measuring Error

- Norms measure distance between vectors
  - The distance between v1 and v2 is $\|v1 - v2\|$
- Imagine we have an approximation x_0 to some true value x So…$x = x_0 + \Delta x$

- The absolute error is Δx
- …for vectors we could use $||\Delta x||$
- Can this tell us if an answer is good?

The Last Two Lectures in 3 Slides

- We want to solve Ax=b for x
- We know there will be some error…the solution will be approximate
- There will be input error in the b vector
  - Give an example….
- We want to know how much the solving method amplifies that error

The Last Two Lectures in 3 Slides

- We need to measure error
  - Define ***relative error*** $||x - x_0||/||x||$
    - What is x? What is x_0?
  - Need a magnitude measure for vectors
    - Define a ***vector norm*** so we can compute $||x||$
  - Define a matrix norm $||A|| = \max||Ax||$ for all x with $||x|| = 1$
    - Upper bound on how much a matrix can stretch a vector
  - Define condition number as the ratio of output error to input error

The Last 2 Lectures in 3 Slides

- We want solve Ax=b for x
- How much does the algorithm amplify error?

$$A(x + \Delta x) = (b + \Delta b)$$

$$\frac{\|\Delta x\|}{\|x\|} \Big/ \frac{\|\Delta b\|}{\|b\|} \leq \|A\| \|A^{-1}\|$$

What about matrix multiplication

$$A x = b$$
input
output

- It's an algorithm…
- It has an input (with error) and an output
- What's the condition number of matrix multiplication

$$A(x+\Delta x) = (b+\Delta b)$$

let $B = A^{-1} \longrightarrow x = Bb$ or $\boxed{Bb = x}$

input

output

$$\frac{\|\Delta b\| / \|b\|}{\|\Delta x\| / \|x\|} \leq \|B\| \, \|B^{-1}\| = \|A^{-1}\| \, \|A\|$$

$$\frac{\text{rel } x}{\text{rel } b} \leq C$$

$$\frac{\text{rel } x}{.1} \leq 20$$

$$\text{rel } x \leq 2$$

Matrix Norms

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

$$\|A\|_1 = \max_{1 \leqslant j \leqslant n} \sum_{i=1}^{m} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leqslant i \leqslant m} \sum_{j=1}^{n} |a_{ij}|$$

Matrix norms are defined in terms of an underlying vector norm

In general matrix p-norms are complicated to compute but a couple of them are easy…

What is the 1-norm computing?

What is the inf-norm computing?

Some useful facts and definitions

- A matrix is A is invertible if there is matrix A^{-1} such that $AA^{-1}=I$
- Non-singular means invertible
- $(AB)^T=B^TA^T$
- $(AB)^{-1}=B^{-1}A^{-1}$
- $(A^{-1})^T=(A^T)^{-1}$

Let's look at an example of solving a system

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = A$$

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= \ -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

$$b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Think in terms of Ax=b

What is A?

What is b?

What is x?

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Let's look at an example of solving a system

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad &= \ -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

Let's think in terms of column vectors.

Write b as a linear combination of column vectors of A.

$$u \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + v \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = b$$

Let's look at an example of solving a system

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad &= \; -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

Let's think in terms of geometry.

Each equation is a plane.
What is a solution to the system?

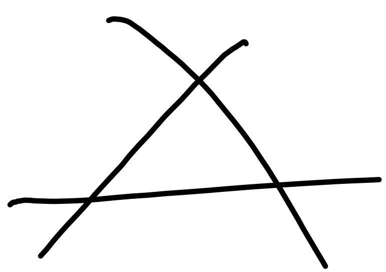
A unique soln is a point

Let’s look at an example of solving a system

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad &= \ -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

Let’s think in terms of geometry.

What are some ways the planes could be arranged so that there are no solutions?

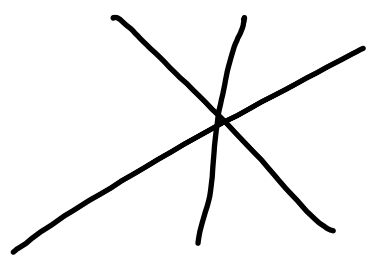


Let's look at an example of solving a system

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

Let's think in terms of geometry.

What are some ways the planes could be arranged so that there are an infinite number of solutions solutions?



Let's solve using Gaussian Elimination

$$2u + v + w = 5$$
$$4u - 6v \quad = -2$$
$$-2u + 7v + 2w = 9$$

$$\begin{matrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{matrix} \xrightarrow[r_1 + r_3]{(-2r_1 + r_2)} \begin{matrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{matrix}$$

$$\xrightarrow{} r_2 + r_3 \qquad \begin{matrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{matrix}$$

Let's solve using Gaussian Elimination

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad &= -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

Backward Substitution

$$
\begin{matrix}
2 & 1 & 1 & 5 \\
0 & -8 & -2 & -12 \\
0 & 0 & 1 & 2
\end{matrix}
$$

① $w=2$

② $-8v = -12 + 4 \leftarrow 2(-2)$

$v = 1$

③ $2u = 5 - 2 - 1 = 2$

$u = 1$

Let's solve using Gaussian Elimination

$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v \quad &= \ -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

Computational Cost

$n \times n$ matrix

- How many operations does it take to get a triangular matrix?

1 operation = a division or a multiplication + subtraction

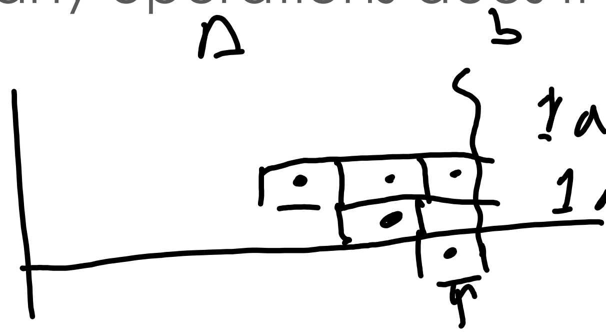
Eliminating 0's in first column for $\underline{\underline{n-1}}$ rows below row $\underline{1}$ we do $\underline{\underline{n}}$ mult/subs

$n(n-1) \rightarrow O(n^2)$ already

Over all columns $\left\{ \sum_{k=1}^{n} k(k-1) = \frac{n^3 - n}{3} = O(n^3) \right.$

Computational Cost

- How many operations does it take to do backward substitution?



1 div + 2 mult/sub
1 div + 1 mult/sub
1 divide

$$\sum_{i=1}^{n} i = O(n^2)$$

$$1 + 2 + \dots + n$$

$$n + (n-1) + \dots + 1$$

$$\underbrace{(n+1) + \dots (n+1)}_{n} = \frac{n(n+1)}{2} = \sum_{i=1}^{n} i$$

Semi-random interesting linear algebra application

- Instancing for complex scenes in computer graphics
- We've seen that matrices can transform points (rotation, scaling, and translation)

https://www.youtube.com/watch?v=bvNtu9SGC9g

https://www.youtube.com/watch?v=ANj3rvkKHAM