CS 357: Numerical Methods

Lecture 4: LU Decomposition

Eric Shaffer

A Quick Word Solving Linear Systems

If A^{-1} exists then

$$Ax = b \qquad \Longrightarrow \qquad x = A^{-1}b$$

but

Do not compute the solution to $Ax = b$ by finding A^{-1}, and then multiplying b by A^{-1}!

We see: $x = A^{-1}b$

We do: **Solve $Ax = b$ by Gaussian elimination or an equivalent algorithm**

Triangular Matrices

The generic lower and upper triangular matrices are

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ l_{n1} & & \cdots & l_{nn} \end{bmatrix}$$

and

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & u_{nn} \end{bmatrix}$$

The triangular systems

$$Ly = b \qquad Ux = c$$

are easily solved by **forward substitution** and **backward substitution**, respectively

Solving Triangular Systems

Solving for $x_1, x_2, \ldots, x_n$ for a lower triangular system is called **forward substitution**.

```
given L, b
x_1 = b_1/ℓ_11
for i = 2...n
   s = b_i
   for j = 1...i-1
      s = s - ℓ_{i,j} x_j
   end
   x_i = s/ℓ_{i,i}
end
```

Using forward or backward substitution is sometimes referred to as performing a **triangular solve**.

Row Echelon Form

- Gaussian Elimination transforms A into row echelon form
- Example

$$\begin{bmatrix} 2 & 4 & 1 & 5 & 7 & 10 & 11 \\ & 3 & 7 & 9 & 1 & 5 & 5 \\ & & & & 5 & 8 & 9 \\ & & & & & 7 & 4 \\ & & & & & & 3 \end{bmatrix}$$

- How is this different from Upper Triangular?
- Can we successfully solve the above system using GE?

Row Echelon Form

- All nonzero rows are above any rows of all zeroes
- The leading coefficient of a nonzero row is always to the right of the leading coefficient of the row above
- So…there can be zeroes on and above the diagonal
- Note that in REF that every row is linear combo of the original rows.

Imagine a world with a lot of b's

- $Ax=b_i$
- Do we need to do full Gaussian Elimination for each new b_i?

Imagine a world with a lot of b's

- $Ax=b_i$
- Do we need to do full Gaussian Elimination for each new b_i?
- No…we can perform GE on A to factor it into A=LU
- How does this let us save work?

Imagine a world with a lot of b’s

- Suppose we are given A=LU
- So LUx=b
- We can solve Ly = b
- How? How much work for an nxn matrix L?
- We can then solve Ux=y
- How? How much work for an nxn matrix L?

Imagine a world with a lot of b's

- So, how do we find factor A into LU?
- Given A, we can generally find an upper triangular matrix
  - Through Gaussian Elimination (GE)
- How do we find L?
  - We can find it if we keep track of what we do in GE
  - We use matrix multiplication to do the elimination....
  - …Elimination Matrices

Elimination Matrices

- Annihilate entries below k^{th} element in a by a transformation:

$$M_k a = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & -m_{k+1} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -m_n & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

  where $m_i = a_i / a_k$, $i = k+1, \dots, n$.
- The divisor a_k is the "pivot" (and needs to be nonzero)

Elimination Matrices

- Matrix M_k is an "elementary elimination matrix": adds a multiple of row k to each subsequent row, with "multipliers" m_i so that the result is zero in the k^{th} column for rows $i > k$.
- M_k is unit lower triangular and nonsingular
- $M_k = I - m_k e_k^T$ where $m_k = [0, \ldots, 0, m_{k+1}, \ldots, m_n]^T$ and e_k is the k^{th} column of the identity matrix I.
- $M_k^{-1} = I + m_k e_k^T$, which means M_k^{-1} is also lower triangular, and we will denote $M_k^{-1} = L_k$.

Can you prove $M_k^{-1} = I + m_k e_k^T$?

Example

Let $a = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$.

$$M_1 a = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

and

$$M_2 a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

Example

So

$$L_1 = M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},\ L_2 = M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$$

which means

$$M_1 M_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 1/2 & 1 \end{bmatrix},\ L_1 L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1/2 & 1 \end{bmatrix}$$

Gaussian Elimination with Elementary Elimination Matrices

- To reduce $Ax = b$ to upper triangular form, first construct M_1 with a_{11} as the pivot (eliminating the first column of A below the diagonal.
- Then $M_1Ax = M_1b$ still has the same solution.
- Next construct M_2 with pivot a_{22} to eliminate the second column below the diagonal.
- Then $M_2M_1Ax = M_2M_1b$ still has the same solution
- $M_{n-1}\dots M_1Ax = M_{n-1}\dots M_1b$
- Let $M = M_nM_{n-1}\dots M_1$; then $MAx = Mb$, with MA upper triangular. Then back solve.

Another Way to Look at A

We've mentioned L and U today. Why?
Consider this

$$A = A$$

$$A = (M^{-1}M)A$$

$$A = (M_1^{-1}M_2^{-1}\dots M_n^{-1})(M_nM_{n-1}\dots M_1)A$$

$$A = (M_1^{-1}M_2^{-1}\dots M_n^{-1})((M_nM_{n-1}\dots M_1)A)$$

But MA is upper triangular, and we've seen that $M_1^{-1}\dots M_n^{-1}$ is lower triangular. Thus, we have an algorithm that factors A into two matrices L and U.