CS 357: Numerical Methods

Lecture 7: LU Applications

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Decomposing into PA=LU

You try it…

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

$$M_1 P_1 A = \begin{bmatrix} 2 & 5 & 8 \\ 0 & -3/2 & -1 \\ 0 & -3/2 & -3 \end{bmatrix}$$

$$\begin{matrix} 2 & 5 & 8 \\ 0 & -3/2 & 1 \end{matrix}$$

PA=LU Example

$$P_2 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PA = LU$$

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

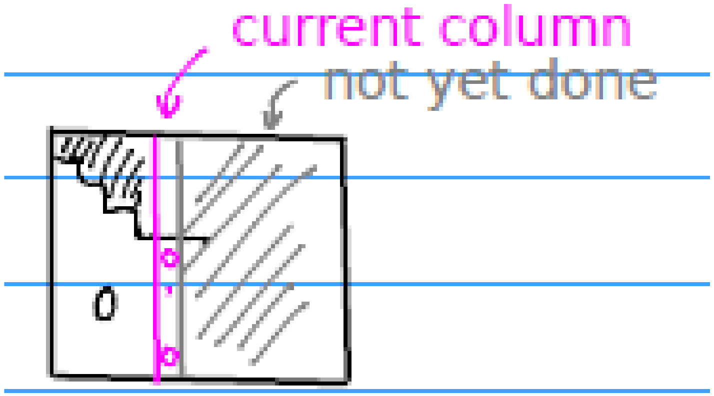
$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

PA=LU Example

PA=LU Example

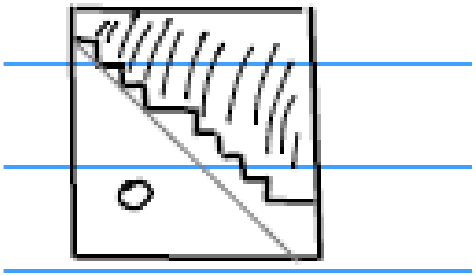
Can LU Fail?

- What happens if the largest below-diagonal value is 0?
- That means the column is already upper-triangular



Pivoted LU Cannot Fail

- You will end up with U in row echelon form



- Which is technically upper triangular

LU on a Singular Matrix

- We still get PA=LU
- P must be invertible (why?)

$P_i^{-1} = P_i^T$

- L must be invertible (why?)

product of P_i's M_i
both invertible

- …means that U must be singular

LU on a Singular Matrix

- All the non-zero rows of U are linearly independent
- Why?

leading 0's

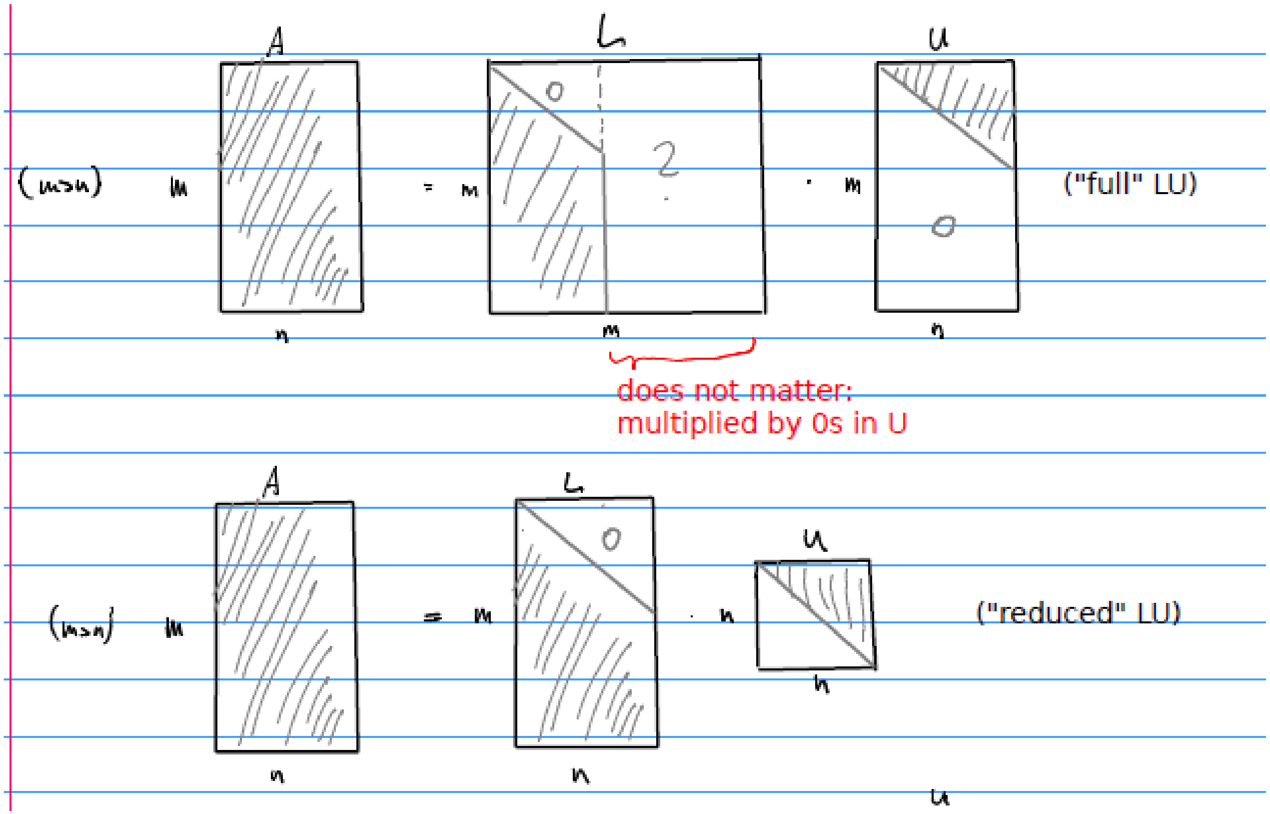
- So what must U look like?

zero rows on bottom

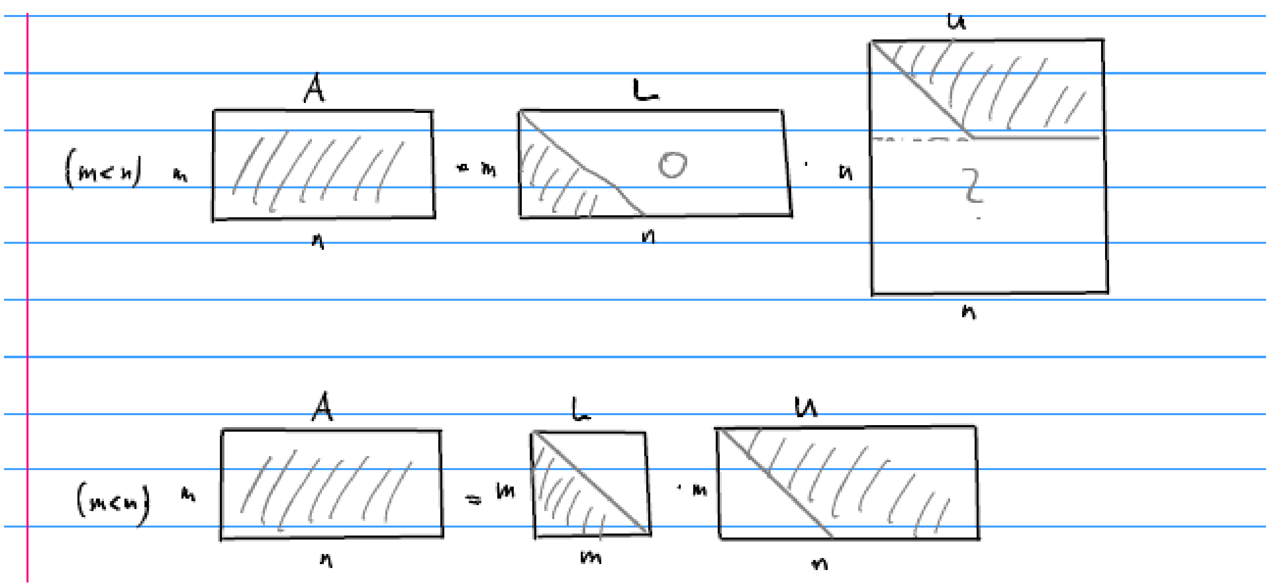
What about LU on non-square matrices?

- For an mxn matrix there are four cases based on
- Is M > N?
- Is the decomposition into one of the following:
  - An mxm times an mxn
  - An mxn times an nxn

What about LU on non-square matrices?



What about LU on non-square matrices?



Full versus Reduced LU

- The reduced version requires less space
  - Doesn't include zero rows in U
  - Doesn't include columns that will be zeroed out in L
- Usually what software will produce…
- Quick quiz question: is LU factorization unique?

No

Applications of LU: Solving Systems of Linear Equations

- Ax=b
- PA=LU
- PAx=Pb
- LUx=Pb
- What comes next?

Solve
$Ly = Pb$
then $Ux = y$

Solving Matrix Equations

- AX=B
- A,X,B are square and same size
- Example:
  $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- Factor A $O(n^3)$
- Then solve for X column by column….
- Computational cost is: n forward subs

$\longrightarrow n\,O(n^2) \rightarrow O(n^3)$

Find the Basis of a Span

- Suppose we are given n possibly linearly dependent vectors
  - x1,x2,x3,..,xn
- Let V be the space spanned by the xi
  - A Span is the set of linear combinations of a set of vectors
- A set of linearly independent vectors spanning V is a Basis

Define	$A = \begin{pmatrix} \text{---} x_1 \text{---} \\ \\ \text{---} x_n \text{---} \end{pmatrix}$
Obtain	$PA = LU$

Non-zero rows of U form a Basis

Compute a Determinant

- Find det(A)
- If we have PA=LU
- det(p)det(A)=det(L)det(U)

→ det(P) det(A) = det(L) det(U)

±1

1

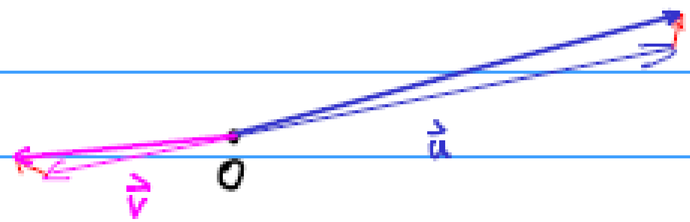
product of diagonal entries

Find the Rank of a Matrix

- The rank of a matrix is the number of linearly independent columns (or rows)
- Testing two vectors for linear dependence on a computer is fraught with peril
- Why? Floating Point

Find the Rank of a Matrix

Suppose we would like to test two inexact vectors for linear dependence.



True: $\vec{u} = \alpha \vec{v}$ (linearly dependent)

Computed: $\vec{u} \neq \alpha \vec{v}$ (not linearly dependent)

<u>Lesson:</u> We cannot hope for exact equality on a computer.
Instead, we must define some sort of tolerance.

Find the Rank of a Matrix

- Taking that into account we can…
- Use LU factorization
  - In exact arithmetic rank-deficiency will generate 0 rows in U
- Use some tolerance since we wont get exact zeros