CS 357: Numerical Methods

Lecture 9:
Rank Finding
Nullspaces
Orthogonality

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Some slides adapted from Linear Algebra by David C. Lay

Rank Finding

- Find A=LU
- Find rank of A by looking at number of non-zero rows in U
- ***Does not work for pivoted LU: PA=LU !***
- ***If you need to pivot you'll need to do something else***
- You can compute an echelon factorization $A=M^{-1}U$
- See demo "LU and upper echelon form"

Finding the Nullspace of A

- The Nullspace of A:

  is the set of vectors x such that $Ax=0$

- Note that if $Ax=b$ and $As=0$ that $A(x+s)=b$

- Does the Nullspace always exist?

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

Let *M* be the matrix for the linear mapping *T*
(ie: $T\overline{x} = M\overline{x}$ **)**

Note:

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This vector is in the **null space** of *T*

The vectors in the null space are the solutions to $M\overline{x} = \overline{0}$

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Let *M* be the matrix for the linear mapping *T*
(ie: $T\overline{x} = M\overline{x}$ **)**

To find a basis for the null space of *T* you must solve:

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$\left(\begin{array}{cccc|c} 2 & 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 3 & 2 & 5 & 2 & 0 \end{array}\right) reduces \quad to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\begin{aligned} w &= -1y \\ x &= -1y \quad -1z \\ y &= 1y \\ z &= \quad\quad 1z \end{aligned}$$

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

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$$\begin{aligned} w &= -1y \\ x &= -1y \quad -1z \\ y &= 1y \\ z &= \qquad\quad 1z \end{aligned}$$

Every vector in the null space looks like:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

Let *M* be the matrix for the linear mapping *T*
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To find a basis for the null space of *T* you must solve:

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A basis for the null space = $\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

Every vector in the null space looks like:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Finding the Nullspace of A

- Andreas will go over this in more detail Tuesday

Inner Products

- Dot product is an example
  - $v \cdot w = \sum v_i w_i$
- A function f with two vector arguments
  With the following properties

$$f(\alpha x, y) = \alpha f(x, y)$$
$$f(x + y, z) = f(x, z) + f(y, z)$$
$$f(x, y) = f(y, x)$$
$$f(x, x) \geq 0$$
$$f(x, x) = 0 \leftrightarrow x = 0$$

Dot Product Applications

- Can be used to measure difference between vectors
  - distance

$$dist_2(w, v) = \sqrt{(w - v) \times (w - v)}$$

  - angle

$$v \times w = \|v\|\|w\| \cos q$$

- …so lots of applications

Orthogonality

- Two vectors x and y are ***orthogonal*** if $x \cdot y = 0$
- We say x is perpendicular to y: $x \perp y$
- In the case of the dot product,
  x and y form a 90 degree angle

Computer Graphics:
Hidden Surface Removal

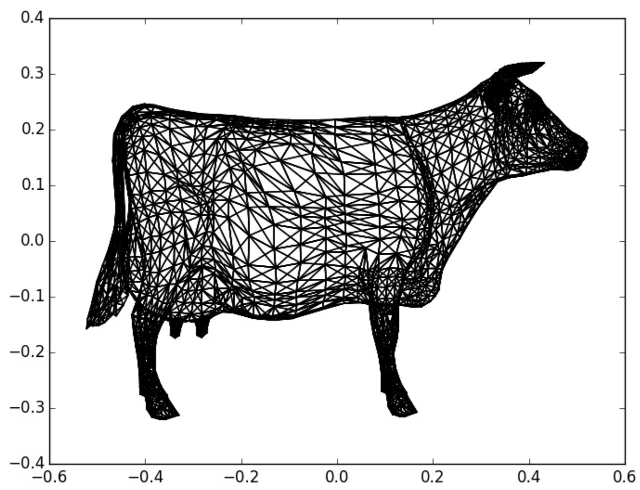
- 3D Models are usually meshes of triangles
- On a surface,
  a triangle facing away from the viewer need not be rendered
  - Why?
- How can I use an inner product to test if a triangle is backfacing?

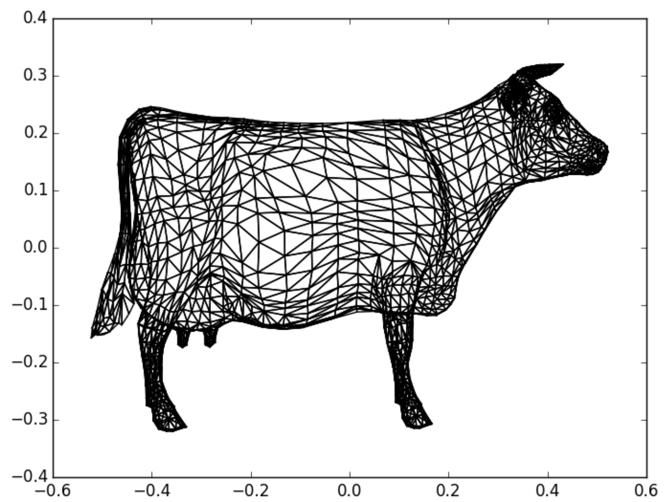
Computer Graphics: Hidden Surface Removal

Computer Graphics:
Hidden Surface Removal

Challenge: Hidden Surface Removal

- See if you can complete the code so that only forward-facing triangles are drawn….





Challenge: Silhouette Rendering

- How can I draw only the lines that are on the silhouette?