

CS 357: Numerical Methods

Lecture 9: Orthogonality

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Some slides adapted from Linear Algebra by David C. Lay

Rank Finding

- Find $A=LU$
- Find rank of A by looking at number of non-zero rows in U
- ***Does not work for pivoted LU: $PA=LU$!***
- ***If you need to pivot you'll need to do something else***
- You can compute an echelon factorization $A=M^{-1}U$
- See demo “LU and upper echelon form”

Finding the Nullspace of A

- The Nullspace of A:

is the set of vectors x such that $Ax=0$

- Note that if $Ax=b$ and $As=0$ that $A(x+s)=b$

- Does the Nullspace always exist?

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

Let M be the matrix for the linear mapping T
(ie: $T\bar{x} = M\bar{x}$)

Note: $\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

This vector is in the null space of T

The vectors in the null space are the solutions to $M\bar{x} = \bar{0}$

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

Let M be the matrix for the linear mapping T
(ie: $T\bar{x} = M\bar{x}$)

To find a basis for the null space of T you must solve:

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$\begin{pmatrix} 2 & 1 & 3 & 1 & | & 0 \\ 1 & 1 & 2 & 1 & | & 0 \\ 3 & 2 & 5 & 2 & | & 0 \end{pmatrix} \text{ reduces to } \begin{pmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

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 (ie: $T\bar{x} = M\bar{x}$)

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$$\begin{pmatrix} 2 & 1 & 3 & 1 & | & 0 \\ 1 & 1 & 2 & 1 & | & 0 \\ 3 & 2 & 5 & 2 & | & 0 \end{pmatrix} \text{ reduces to } \begin{pmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$w = -1y$
 $x = -1y - 1z$
 $y = 1y$
 $z = 1z$

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

Let M be the matrix for the linear mapping T
 (ie: $T\bar{x} = M\bar{x}$)

To find a basis for the null space of T you must solve:

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$$\begin{pmatrix} 2 & 1 & 3 & 1 & | & 0 \\ 1 & 1 & 2 & 1 & | & 0 \\ 3 & 2 & 5 & 2 & | & 0 \end{pmatrix} \text{ reduces to } \begin{pmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\begin{aligned} w &= -1y \\ x &= -1y - 1z \\ y &= 1y \\ z &= 1z \end{aligned}$$

Every vector in the null space looks like:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix}$$

Let M be the matrix for the linear mapping T
 (ie: $T\bar{x} = M\bar{x}$)

To find a basis for the null space of T you must solve:

$$\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 5 & 2 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A basis for the null space =

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Every vector in the null space looks like:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Finding the Nullspace of A

- Andreas will go over this in more detail Tuesday

Inner Products

- Dot product is an example

- $v \cdot w = \sum v_i w_i$

- A function f with two vector arguments
With the following properties

$$f(\alpha x, y) = \alpha f(x, y)$$

$$f(x + y, z) = f(x, z) + f(y, z)$$

$$f(x, y) = f(y, x)$$

$$f(x, x) \geq 0$$

$$f(x, x) = 0 \leftrightarrow x = 0$$

Dot Product Applications

- Can be used to measure difference between vectors

- distance

$$\mathit{dist}_2(w, v) = \sqrt{(w - v) \times (w - v)}$$

- angle

$$v \times w = \|v\| \|w\| \cos q$$

- ...so lots of applications

Orthogonality

- ▣ Two vectors x and y are **orthogonal** if $x \cdot y = 0$
- ▣ We say x is perpendicular to y : $x \perp y$
- ▣ In the case of the dot product, x and y form a 90 degree angle

Computer Graphics: Hidden Surface Removal

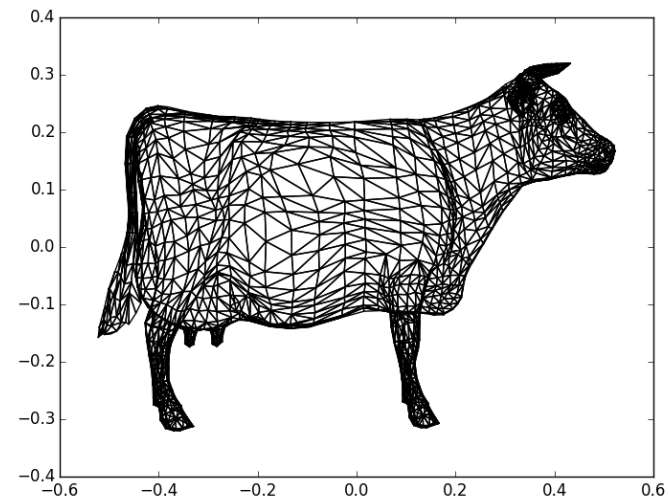
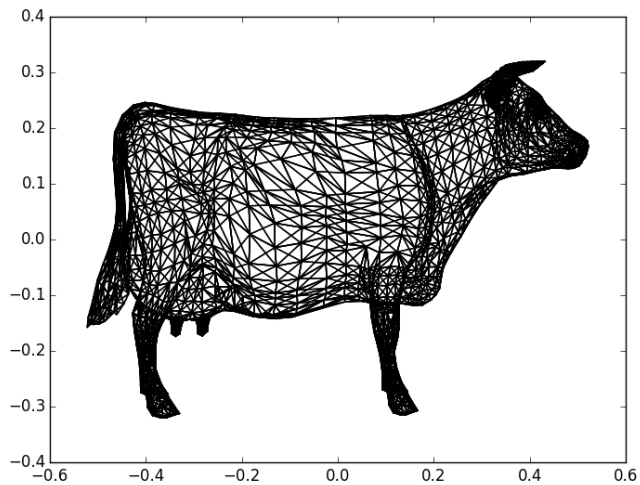
- 3D Models are usually meshes of triangles
- On a surface,
a triangle facing away from the viewer need not be rendered
 - Why?
- How can I use an inner product to test if a triangle is backfacing?

Computer Graphics: Hidden Surface Removal

Computer Graphics: Hidden Surface Removal

Challenge: Hidden Surface Removal

- See if you can complete the code so that only forward-facing triangles are drawn....



Challenge: Silhouette Rendering

- How can I draw one the lines that are on the silhouette?

