CS 357: Numerical Methods

Interpolation Error
Piecewise Polynomial Interpolation

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Review: Quadratic Interpolation (using the monomial basis)

t	0	2	3
y	7	4	4

$$f(t) = x_3 t^2 + x_2 t + x_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}$$

$$x = \begin{bmatrix} 7 \\ -2.5 \\ 0.5 \end{bmatrix}$$

Demo: Interpolation with Vandermonde Matrices

- Things to notice:
  - The points that we choose to interpolate at are called ***nodes***
- In this example the interpolation is quadratic
  - But the coefficients are found by solving a linear system
- Derivative of the interpolant exhibits more error than interpolant

Demo: Interpolation Error I

- Assume the function f being interpolated in smooth
  - …it has many derivatives…
- The length ***h*** of the interpolation interval is “sufficiently small”
- Then we have:

interpolant of degree n

original

$$\underbrace{\left|f(x) - \tilde{f}(x)\right|} \leq \underbrace{C h^{n+1}}$$

constant

Error depends on the interval h (the “step-size”)

Interpolation Error

$\text{Error}(h) \approx C h^3$

- Suppose we have
  - A quadratic interpolant
  - An error E
  - An interval length h
- What happens if I use h/2 instead of h?

$$\text{Error}(h/2) \approx C\,(h/2)^3$$
$$= \tfrac{1}{8} C h^3$$

Demo: Interpolation Error II

If a method has an error bound of

$$E \leq Ch^p$$

It is called ***pth order convergent***

So polynomial interpolation with a polynomial of degree n is (n+1)th order convergent.

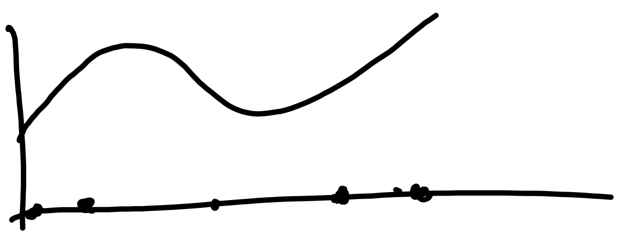
Why does a shorter interval seem to decrease error?

Interpolation Parameters

- What can we choose when interpolating?

Which basis we use
(e.g. monomial)

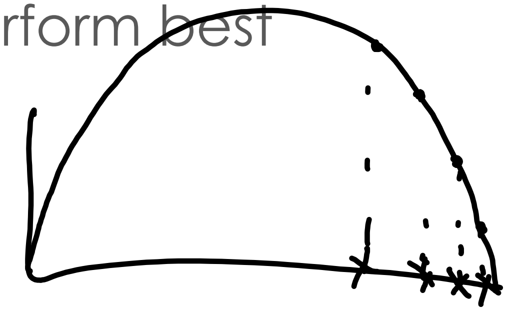
Maybe positions of nodes



Choosing Nodes for Interpolation

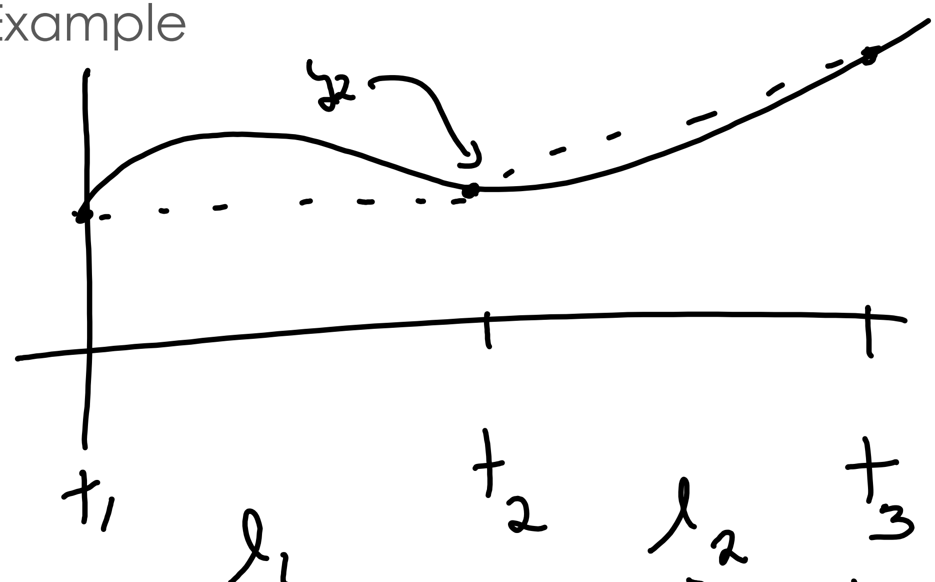
- Demo: Choice of interpolation nodes
- Best if nodes cluster towards the ends of the interval

- On [-1,1] the ***Chebyshev nodes*** perform best
  - $x_k = \cos(\frac{2k-1}{2n}\pi)$ for k=1,…,n



Piecewise Polynomial Interpolation

Example



$l_1 : x_1 + x_2 t = y$

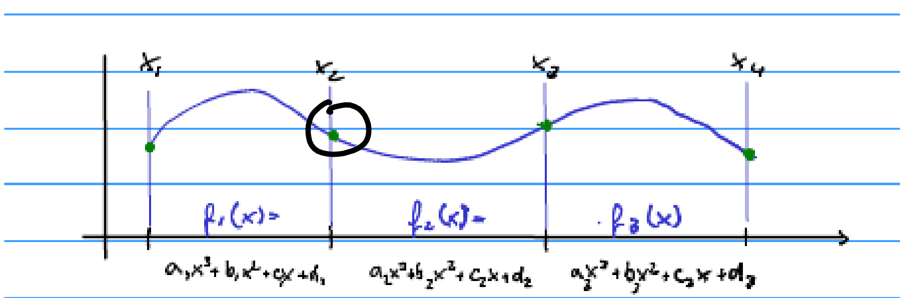
$l_2 : x_3 + x_4 t \; y$

$$\begin{bmatrix} 1 & t_1 & 0 & 0 \\ 1 & t_2 & 0 & 0 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 1 & t_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_3 \end{bmatrix}$$

Example

Cubic Splines

- A ***spline*** is a piecewise polynomial of degree k that is continuously differentiable k-1 times

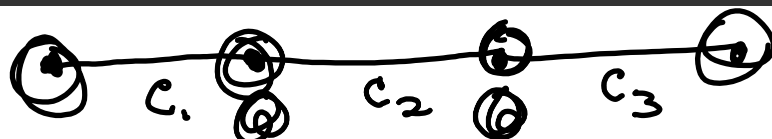


For cubic spline:

- 1st and 2nd derivatives must match at interior points

$$p(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3$$

Cubic Splines: Parameters and Constraints



$n=4$

- Suppose there are n knots (including endpoints)
- How many parameters?

  $4(n-1) = 4n-4$

- How many equations from interpolation constraint?

  $2(n-1)$

- How many equations from 1st derivative match?

  $n-2$ e.g. $c_1'(t_1) = c_2'(t)$ or $c_1'(t_1) - c_2'(t) = 0$

- How many from second derivative match?

  $n-2$

$4n-6$

Cubic Splines: Parameters and Constraints

- 2 free parameters…you could
  - Specify first derivative at t_1 and t_n endpoints
  - Set second derivative to 0 at at t_1 and t_n endpoints
    - This is a ***natural spline***
  - Set first and second derivatives equal at t_1 and t_n
    - Periodic

Example

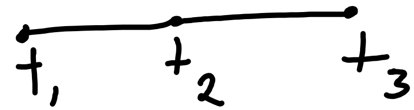
Natural cubic spline on 3 data points

$$p_1(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \alpha_4 t^3$$

$$p_2(t) = \beta_1 + \beta_2 t + \beta_3 t^2 + \beta_4 t^3$$

$$\left.\begin{aligned} p_1(t_1) &= y_1 \\ p_1(t_2) &= y_2 \\ p_2(t_2) &= y_2 \\ p_2(t_3) &= y_3 \end{aligned}\right\}$$

4 equations



Example

Derivatives at t_2

$p_1'(t)$

$$\alpha_2 + 2\alpha_3 t_2 + 3\alpha_4 t_2^2 =$$

$$⑤ \quad \underbrace{\beta_2 + 2\beta_3 t_2 + 3\beta_4 t_2^2}_{\text{1st deriv. of } p_2}$$

$$⑥ \quad 2\alpha_3 + 6\alpha_4 t_2 = 2\beta_3 + 6\beta_4 t_2$$

Natural Spline:

$$⑦ \quad 2\alpha_3 + 6\alpha_4 t_1 = 0$$

$$⑧ \quad 2\beta_3 + 6\beta_4 t_3 = 0$$

Example

1st deriv. match at t_2
2nd " " " "

interpolating constraint

$$
\begin{bmatrix}
1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 \\
1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & t_2 & t_2^2 & t_2^3 \\
0 & 0 & 0 & 0 & 1 & t_3 & t_3^2 & t_3^3 \\
0 & 1 & 2t_2 & 3t_2^2 & 0 & -1 & -2t_2 & -3t_2^2 \\
0 & 0 & 2 & 6t_2 & 0 & 0 & -2 & -6t_2 \\
0 & 0 & 2 & 6t_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 6t_3
\end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}
=
\begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

"natural" spline constraint