Numerical Methods (CS 357)

Worksheet

Part 1. Objectives

- **Long-term goal:** How do I fit a model to data?
- How does QR factorization work? What special cases are there?
- How do I solve (in a least-squares sense) an overdetermined linear system using QR?
- Why should I use QR and not the normal equations $A^TAx = A^Tb$ for least-squares problems?

Part 2. Uniqueness of the least-squares solution

Consider the least-squares problem $Ax \cong b$ where A is tall-and-skinny (i.e. $m \times n$ with $m > n$) and has a nontrivial nullspace (i.e. there is a vector $n \neq 0$ so that $An = 0$).

If you have a solution x to $Ax \cong b$, can you find another solution x' (so that $\|Ax - b\|_2 = \|Ax' - b\|_2$)?

(A) No

(B) Yes

Part 3. Norms and Matrices

Given

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

what is $\|Q^Tb\|_2^2$?

Part 4. Least-Squares Residual

Given a QR factorization $A = QR$ with $Q = I$, what is the **square of the 2-norm** of the residual of solving the least-squares problem $Ax \cong b$ going to be if

$$b = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}?$$

Part 5. Solving least-squares problems

You are given a matrix A and a right-hand side vector b.

Use a QR factorization of A (from `scipy.linalg.qr`) to solve the least-squares problem $Ax \cong b$.

Hints:

- `scipy.linalg.qr` returns the full QR factorization by default.
- Use `scipy.linalg.solve_triangular`.

INPUTS: `A` and `b`

OUTPUT: `x`

```
import scipy.linalg as la

m, n = A.shape
```