biggest

$$A = \underbrace{\begin{pmatrix} | & | & | & | \end{pmatrix}}_{U} \cdot \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \cancel{\sigma_3} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}}_{V^T}$$

$$= u_1 \cdot \sigma_1 v_1^T + u_2 \sigma_2 v_2^T$$

$$\approx \underbrace{\begin{pmatrix} | & | & | \end{pmatrix}}_{U} \cdot \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \cdot \begin{pmatrix} - \\ - \\ - \end{pmatrix}$$

④ Eigenvalue problems

$Ax = \lambda x \qquad x \neq 0$

eigenvalue λ
eigenvector x

Spectrum, spectral radius

Finding:

λ eigenvalue $\Leftrightarrow (A - \lambda I)x = 0$ solvable for $x \neq 0$
$\Leftrightarrow A - \lambda I$ singular
$\Leftrightarrow \det(A - \lambda I) = 0$

char. poly of deg. n

has n roots (possibly in $\mathbb{C}$)

for $n \geq 5$: must approximate!

$Ax = \lambda x \leftarrow$ right eigenvector
$y^T A = \lambda y^T$

Multiplicity:

Algebraic $\longleftrightarrow$ Geometric

multipl. as roots of CP $\geq$ lin. indep. eigenvectors

$(\lambda-\alpha)^{n_1} \ldots$ $\quad$ " $>$ " $\rightarrow$ defective

Example $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad$ CP: $(1-\lambda)^2$

Eigv: 1×2

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\cdot\begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{matrix} x+y=x \\ y=0 \end{matrix}$$

eigenvectors: $\begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \setminus \{0\}$

A not defective: A diagonalizable

$$X = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \quad \text{eigenvectors}$$

$$A = XDX^{-1}$$

similarity transf

go to eigenb.

stretch

go to orig. basis

4.2 Sensitivity

Assume A not defective

Suppose $X^{-1}AX = D$

Perturb $A \to A+E$

$$X^{-1}(A+E)X = D + \underbrace{X^{-1}EX}_{F}$$

Suppose v is a perturbed eigenvector,

$$(D+F)v = \mu v$$

$$Fv = (\mu I - D)v \qquad (\mu I - D)^{-1}$$

μ is not eigenvalue of A

$$(\mu I - D)^{-1} F v = v$$

$$\Rightarrow \|(\mu I - D)^{-1}\| \, \|F\| \cdot \cancel{\|v\|} \geq \cancel{\|v\|}$$

$$\underbrace{\|(\mu I - D)^{-1}\|^{-1}}_{} \leq \|F\|$$

diagonal:
biggest entry on diagonal
(closest eigenvalue of A to μ)

λ_k

$$|\mu - \lambda_k| = \|(\mu I - D)^{-1}\|^{-1} \leq \|F\|$$
$$\leq \|X\| \|E\| \|X^{-1}\|$$
$$= \text{cond}_2(X) \, \|E\|$$