$|\lambda_k - \mu| \leq \mathrm{cond}(X) \; \|E\|_2 \qquad A \to A + E$

4.3 Properties and Transformations

Suppose $Ax = \lambda x \quad (x \neq 0)$.

Shift $\quad A - \sigma I \qquad (A - \sigma I)x = (\lambda - \sigma I)x$

Invert $\quad A^{-1} \qquad A^{-1}x = \frac{1}{\lambda}x$

Power $\quad A^k \quad k \in \mathbb{Z} \qquad A^k x = \lambda^k x$

Polynomial $\quad aA^2 + bA + cI \qquad (\dots)x = (a\lambda^2 + b\lambda + c)x$

Similarity $\quad T^{-1}AT \qquad y := T^{-1}x$

non-sing. $\qquad T^{-1}ATy = \lambda y$

$(A^*)^T$

Orthonormal $A^TA=I$ $A^HA=I$ (unitary)

$|\lambda|=1$

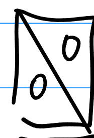
Symmetric $A^T=A$ $A^H=A$ (hermitian)

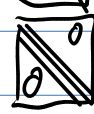
real eigenvectors

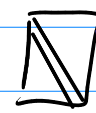
Normal $A^HA=A\,A^H$

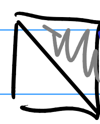
orth eigenvectors

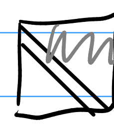
✓ can read off eigenvalues

Diagonal  ✓

Tridiagonal 

Bidiagonal  ✓

Triangular ("Schur")  ✓

Hessenberg 

$\det(A-\lambda I)$

Tod: Schur form

Every matrix is orthonormally similar to an upper triangular matrix:

$$A = \underbrace{Q}_{\text{orth.}} \underbrace{U}_{\text{upper triang.}} Q^T$$

Existence: (assume non-defective)

$$Av = \lambda v \quad (v \neq 0)$$
$$V = \text{span}\{v\}$$

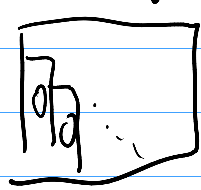
$$A: \quad V \longrightarrow V$$
$$V^\perp \longrightarrow V \oplus V^\perp$$

orth. complement

$V^\perp = \{ w : w \perp V \}$

Matrix form:

$$A = \underbrace{\begin{pmatrix} v & \begin{matrix}\text{Basis}\\ \text{of}\\ V^\perp\end{matrix} \end{pmatrix}}_{Q_1} \begin{array}{cc} \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix} & \begin{array}{l} \}\,V \\ \\ V^\perp \end{array} \\ \begin{array}{cc} V & \quad V^\perp \end{array} & \end{array} Q_1^T$$



4.4 Computing eigenvalues

Eigenvalues of $A^{20,000}$?

$\left(|\lambda_1| > |\lambda_2| > | \dots \right.$

$\uparrow x_1 \quad \uparrow x_2$

$A^{20,000}(\alpha x_1 + \beta x_2)$

$= \alpha \lambda_1^{20,000} x_1 + \beta \lambda_2^{20,000} x_2$

$: \lambda_1^{20,000}$

$\alpha x_1 + \beta \underbrace{\frac{\lambda_2^{20,000}}{\lambda_1^{20,000}}} x_2$

$\left(\frac{\lambda_2}{\lambda_1}\right) < 1$

$\Rightarrow \left(\frac{\lambda_2}{\lambda_1}\right)^{20,000} << 1$

Power iteration

Problems: $\rightarrow$ Overflow $\leftarrow$ Fix: normalize

$\hookrightarrow \lambda_1 = \lambda_2$

Normalized power iteration

Convergence of power it.?

$e_k = \| \lambda_1 - \hat{\lambda}_1^{(k)} \|$ $\qquad e_{k+1} = \underbrace{\frac{|\lambda_2|}{|\lambda_1|}}_{<1} \cdot e_k$

Shift: $A - \sigma I$

small!

$$e_{k+1} = \underbrace{\frac{|\lambda_2 - \sigma|}{|\lambda_1 - \sigma|}}_{\text{small!}} \cdot e_k$$

big!

Which σ?
Hard to tell! $\sigma = \lambda_2$?

Don't know eigenvalues to plug in.

→ Shifts not very useful in (fwd) power it.

→ Can we estimate eigenvalues?

$\frac{\|Ax\|}{\|x\|}$ → no sign

$x^T \underbrace{A^T A}_{\text{Sml}} x$

Rayleigh quotient:

$\lambda \approx \frac{x^T A x}{x^T x}$ $\qquad$ If x is eigenv. $\Rightarrow \frac{x^T \lambda x}{x^T x} = \lambda$