

4.5 Krylov space methods

QR it: $\text{span} \{ A^l y_1, \dots, A^l y_k \}$

Krylov: $\text{span} \{ \underbrace{x_0}_{x_0}, \underbrace{Ax_0}_{x_1}, \underbrace{A^2 x_0}_{x_2}, \dots, \underbrace{A^{k-1} x_0}_{x_{k-1}} \}$

Define $K_k := [x_0 \ x_1 \ x_2 \ \dots \ x_{k-1}] \leftarrow n \times k$

$AK_n = [x_1 \ x_2 \ \dots \ x_n]$
 $= K_n [e_2 \ e_3 \ e_4 \ \dots \ e_n k x_n] \quad e_2 = (0, 1, 0, \dots)$

$\underline{K_n^{-1} A K_n} = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \leftarrow \text{upper Hessenberg}$

$Q_n K_n = K_n \rightarrow Q_n = K_n R_n^{-1} \rightarrow Q_n^{-1} = R_n K_n^{-1}$

$Q_n^T A Q_n = Q_n^{-1} A Q_n$
 $= R_n \underbrace{K_n^{-1} A K_n}_{\text{Hessenberg}} R_n^{-1}$
 $= \begin{pmatrix} \triangleright & & \\ & \triangleright & \\ & & \triangleright \end{pmatrix} =: H$

Fact:

$\begin{matrix} \triangleright & \triangleright & \triangleright \\ \triangleright & \triangleright & \triangleright \end{matrix}$

$A Q_n = Q_n H$

$\xrightarrow{q_i^T \cdot}$
 q_i^T

$\underline{q_i^T A q_j} = H_{ij}$

$$Q_n = \left(\begin{array}{c|c} \text{computed} & \text{not yet computed} \\ \hline \end{array} \right)$$

$$H = Q^T A Q = \left(\begin{array}{c|c} \text{computed} & \\ \hline \end{array} \right) A \left(\begin{array}{c|c} \text{computed} & \\ \hline \end{array} \right) = \left(\begin{array}{c|c} \text{computed} & \\ \hline \end{array} \right) \left(\begin{array}{c|c} \text{already computable using} & \\ \hline \text{eigenvalues?} & \text{current } Q \end{array} \right)$$

"Ritz values"

expensive? \rightarrow simply restart w/ new initial vect.

symmetric? \rightarrow Lanczos iteration

⑤ Nonlinear equations

Develop machinery $f(\vec{x}) = \vec{0}$ $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$f(\vec{x}) = \vec{y}$$

$$\hat{f}(\vec{x}) := f(\vec{x}) - \vec{y}$$

$$\hat{f}(\vec{x}) = \vec{0}$$

$$f(\dots) = 0$$

$$f(\dots) = \vec{0}$$