Announcements

\[ x_{k+1} x_{k+2} x_{k+3} \ldots \]

Error: \( e_k = x_k - x^* \)

\[ \| e_{k+1} \| \leq C \| e_k \|^n \Rightarrow \text{asymptotic, helpful if } \| e_k \| \text{ is small} \]

\[ \frac{\| e_{k+1} \|}{\| e_k \|^n} \leq C \]

\[ \Rightarrow 0 \text{ if } n \text{ is high enough} \]

\[ 0 < C_{m+1} \leq \frac{\| e_{m+1} \|}{\| e_m \|^n} \leq C_{m+1} \]

\[ \lim_{n \to \infty} \frac{\| e_{m+1} \|}{\| e_m \|^n} = C > 0 \]

\( r = 1 \text{ linear} \)
\( r = 2 \text{ quadratic} \)
5.3  Stopping iterative procedures

\[ f(x_k) \approx 0 \]

If \( \| x_k \| < \text{tol} \)

\[ \| x_k - x_{k+1} \| < \text{tol} \]

\[ \| x_k - x_{k+1} \| / \| x_k \| < \text{tol} \]

5.4  Methods in 1D

Bisection

Rate of conv: 1

Constant: \( \frac{1}{2} \)
Fixed point iteration

\[ x_0 = \text{(starting guess)} \]

\[ x_{k+1} = g(x_k) \]

When does FPI converge?

Assume:

- \( g \) smooth

- \( |g'(x)| < 1 \)

Error:

\[ e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*) \]

Mean value theorem

\[ e_{k+1} = g(x_k) - g(x^*) = g'(\theta)(x_k - x^*) = e_k \]

\[ \theta \text{ sufficiently near } x^* \]

\[ \|e_{k+1}\| \leq C \cdot \|e_k\| \]

FPI is (at least) linearly conv. if \( |g'(x)| < 1 \).
\[ g(x^*) = x^* \quad (g'(x^*) = 0) \]

\[ g(x_0) - g(x^*) = g''(\xi) \frac{(x_0 - x^*)^2}{2} \quad \text{(Taylor)} \]

\[
\text{then:} \quad \| x_{k+1} - x^* \| \leq C \| x_k - x^* \|^2
\]

\[ \rightarrow \quad \text{quadratic} \]

WS 17 p1

Newton's method

\[ f(x^*) = 0 \quad R_1 \]

\[ f(x + h) = f(x) + h \cdot f'(x) = 0 \]

\[ h = -\frac{f(x)}{f'(x)} \]

Newton:

\[ x_0 = \text{(starting guess)} \]

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \]

\[ g(x) \]
\[ g'(x) = \frac{f(x)f''(x)}{f'(x)^2} \]

\[ 1 - \frac{c(x)}{b'(x)} = \frac{b(x) - c(x)}{b'(x)} \left( \frac{a(x)}{b(x)} \right) = \frac{a(x)b'(x) - b(x)b'(x)}{b'(x)^2} \]

\[ f(x^*) = 0 \quad f'(x^*) \neq 0 \quad g'(x^*) = 0 \]

simple root at \( x^* \)

quadr. conv. of FPI

Drawback: need derivative

Secant method

\[ f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \]