Announcements.

$x_{k+1} \quad x_{k+2} \quad x_{k+3} \ldots$

Error: $e_k = x_k - x^*$

$$\|e_{k+1}\| \le c \cdot \|e_k\|^r \longrightarrow \text{asymptotic}$$

helpful if $\|e_k\|$ is small

$$\frac{\|e_{k+1}\|}{\|e_k\|^r} \le c$$

$\to 0$ if r not high enough

$$0 < c_{low} \le \frac{\|e_{k+1}\|}{\|e_k\|^r} \le c_{high}$$

$$\Uparrow$$

$$\lim_{k\to\infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = c > 0$$

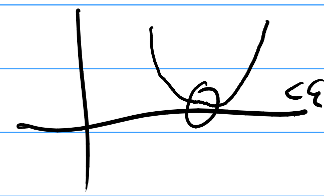
$r=1$ linear

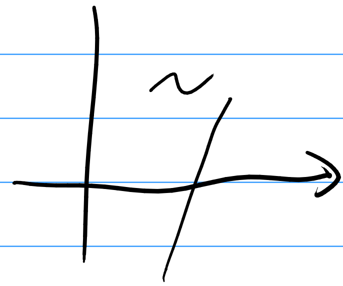
$r=2$ quadratic

5.3 Stopping iterative procedures

x_1, x_2, x_3

$f(x_k) \approx 0$





$\|f(x_k)\| < tol$

$\|x_k - x_{k+1}\| < tol$

$\|x_k - x_{k+1}\| / \|x_{k+1}\| < tol$

5.4 Methods in 1D

Bisection

Rate of conv: 1

Constant: $\frac{1}{2}$

Fixed point iteration

$x_0 =$ (starting guess)

$x_{k+1} = g(x_k)$

When does FPI converge?

Assume:

- g smooth
- $|g'(x^*)| < 1$

Error:

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$

Mean value theorem

$$e_{k+1} = g(x_k) - g(x^*) = \underbrace{g'(\theta)}_{\substack{\text{if sufficiently near } x^* \\ \leq C < 1}} \underbrace{(x_k - x^*)}_{= e_k}$$

$$\|e\|_{k+1} \leq C \cdot \|e_k\|^1$$

FPI is (at least) linearly conv. $|g'(x^*)| < 1$.

If

$g(x^*) = x^*$ $\quad$ $g'(x^*) = 0$

$$g(x_k) - g(x^*) = g''(\xi)\frac{(x_k - x^*)^2}{2} \quad \text{(Taylor)}$$

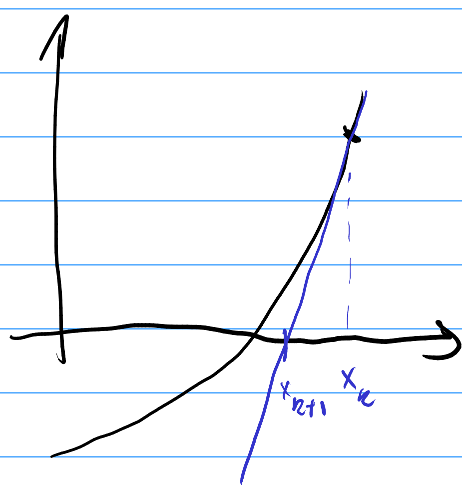
then: $\|e_{k+1}\| \le C\|e_k\|^2$

$\Rightarrow$ quadratic

WS 17 p1

Newton's method

$f(x^*) = 0$ $\quad$ f'



$$f(x+h) = f(x) + h \cdot f'(x) = 0$$

$$h = -\frac{f(x)}{f'(x)}$$

Newton:

$x_0 =$ (starting guess)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad g(x)$$

$$g'(x) = \frac{f(x)\,f''(x)}{f'(x)^2}$$

$$a(x) \qquad b(x)$$

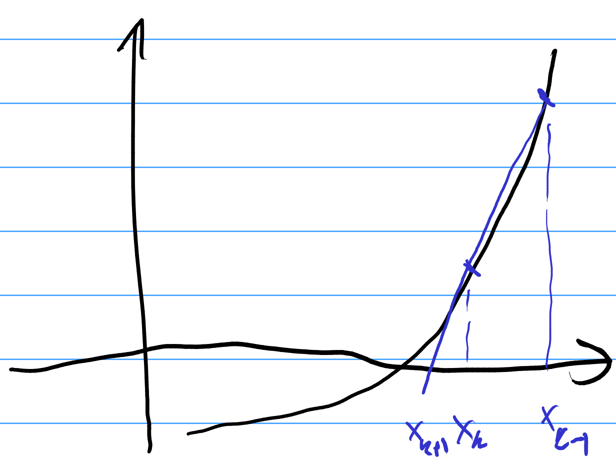
$$1 - \frac{c(x)}{b^2(x)} = \frac{b^2(x)}{b^2(x)} - \frac{c(x)}{b^4(x)} \left(\frac{a(x)}{b(x)}\right)' = \frac{\pm a'(x)b(x) \mp a(x)b'(x)}{b^2(x)}$$

?

$$\underbrace{f(x^*) = 0 \qquad f'(x^*) \neq 0}_{\text{simple root at } x^*} \qquad \underbrace{g'(x^*) = 0}_{\text{quadr. conv. of FPI.}}$$

Drawback: need derivative

Secant method



$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$