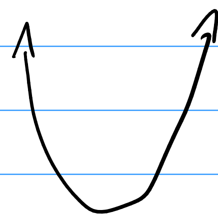
Office hours moved to 3pm

$f: \mathbb{R}^n \to \mathbb{R}$

$\min_{x} f(x)$

$f$ coercive

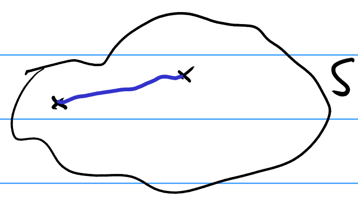


$$\lim_{\|x\| \to \infty} f(x) = +\infty$$

$\Rightarrow$ $f$ global minimum

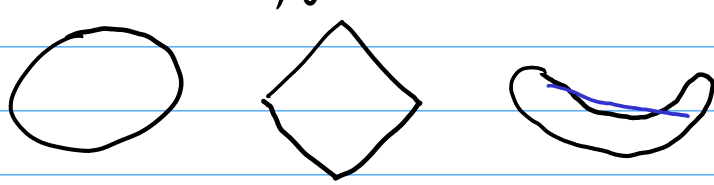
$f$ convex

$S$ convex




For all $x, y \in S$, for all $\alpha \in [0,1]$

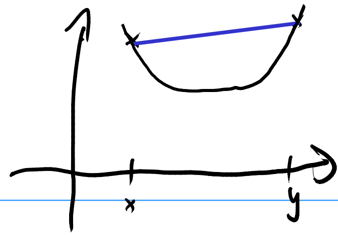
$$\alpha x + (1-\alpha) y \in S$$



$f$ convex on $S \subseteq \mathbb{R}^n$ convex if for all $x, y \in S$, $\alpha \in [0,1]$

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$ "convex"

$<$ "strictly convex"




$f$ convex $\Rightarrow$ each local minimum is a global min

$f$ strictly convex $\Rightarrow$ each local min is a <u>unique</u> global

$f$ convex $\Rightarrow$ $f$ continuous

"regularization" $\leftarrow$ help coercivity and convexity.

## (6.4) Optimality conditions

1D    necessary $f'(x^*) = 0 \Rightarrow x^*$ is extremal pt.

      sufficient $f'(x^*) = 0,\ f''(x^*) > 0$

nD    necessary $\nabla f(x^*) = 0$

      sufficient $\nabla f(x^*) = 0,\ H_f(x^*)$ pos. def.

Hessian matrix

$$H_f(x) = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2}{\partial x_n \partial x_1} & & \frac{\partial^2}{\partial x_n^2} \end{pmatrix} f$$

symmetric

$$f(x+s) = f(x) + \underbrace{\nabla f(x)^T}_{0} s + \frac{1}{2} s^T \underbrace{H_f(x)}_{\geq 0} s$$

Test for PD: use Cholesky    WS 19 p1

## 6.4 Sensitivity and Conditioning

1D: $f(\underbrace{x+h}_{\hat{x}}) = f(\underbrace{x}_{x^*}) + \cancel{f'(x)h} + f''(x)\frac{h^2}{2} + O(h^3)$

Now suppose $x$ is a minimizer

Also suppose $\underbrace{|f(\hat{x}) - f(x^*)|}_{\sim f''(\hat{x})\frac{h^2}{2}} \leq tol$

$f(\hat{x})$ found, $f(x^*)$ actual

$$|x^* - \hat{x}| \leq \sqrt{2\,tol / f''(x^*)}$$

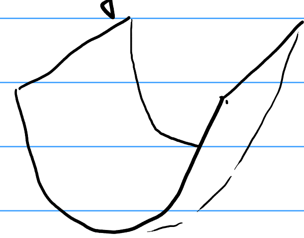
10W: Half as many digits in minimizer as in function values

(Can do better if $\nabla f$ used)

nD: $\underset{\|s\|=1}{f(x^* + hs)} = f(x^*) + \cancel{h \nabla f(x^*)^T s} + \frac{h^2}{2} s^T H_f(x^*) s + O(h^3)$

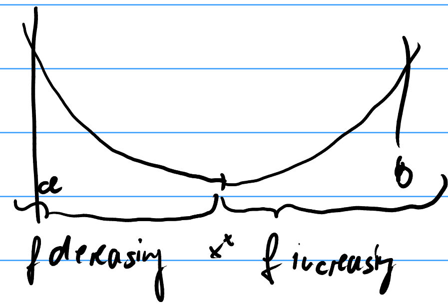
$$\|h\|^2 \leq \frac{2\varepsilon}{\lambda_{min}(H_f(x^*))}$$

Poorly conditioned $H_f$ →



## 6.5 Methods in 1D (unconstrained)

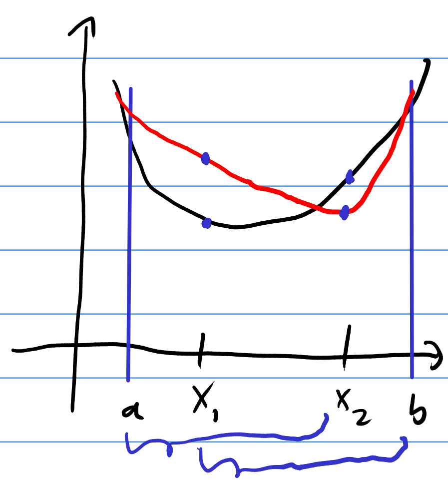
New on assumption on $f$: "unimodal"




$f$ unimodal $\iff$ for all $x_1 < x_2$

- $x_2 < x^* \Rightarrow f(x_1) > f(x_2)$
- $x^* < x_1 \Rightarrow f(x_1) < f(x_2)$

### Golden Section Search




- Pick $x_1, x_2$
- if $f(x_1) > f(x_2)$ start over on $[a, x_2]$
- if $f(x_1) \leq f(x_2)$ start over on $[x_1\ b]$