8 Int and Diff.

8.1 Num. Integration

a, b, f approximate $\int_a^b f(x)\,dx$

Existence, uniqueness: f integrable $\Rightarrow \int f$ exists, unique

pw continuous, bounded $\Rightarrow f$ integrable

Conditioning

$$\hat{f}(x) := f(x) + e(x)$$

$$\left| \int_a^b f(x)\,dx - \int_a^b \hat{f}(x)\,dx \right|$$

$$= \left| \int_a^b e(x)\,dx \right|$$

$$\leq \int_a^b |e(x)|\,dx$$

$$\leq (b-a) \underbrace{\|e\|_\infty}_{\max_{x \in [a,b]} |e|}$$

abs. condition number

8.2 Quadrature methods

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \underbrace{w_i}_{\text{weights}} \, f(\underbrace{x_i}_{\text{nodes}})$$

Idea 1: "interpolatory quadrature"

$$f(x) \approx \sum_{i=1}^{n} f(x_i)\, \underbrace{\ell_i(x)}_{\text{Lagrange basis}}$$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i) \underbrace{\int_a^b \ell_i(x)\,dx}_{\text{too hard.}}$$

1. w/ monomials and (mostly) equispaced points

   "Newton-Cotes quadrature"

2. w/ Chebyshev polys at Chebyshev nodes

   "Clenshaw-Curtis quadrature"

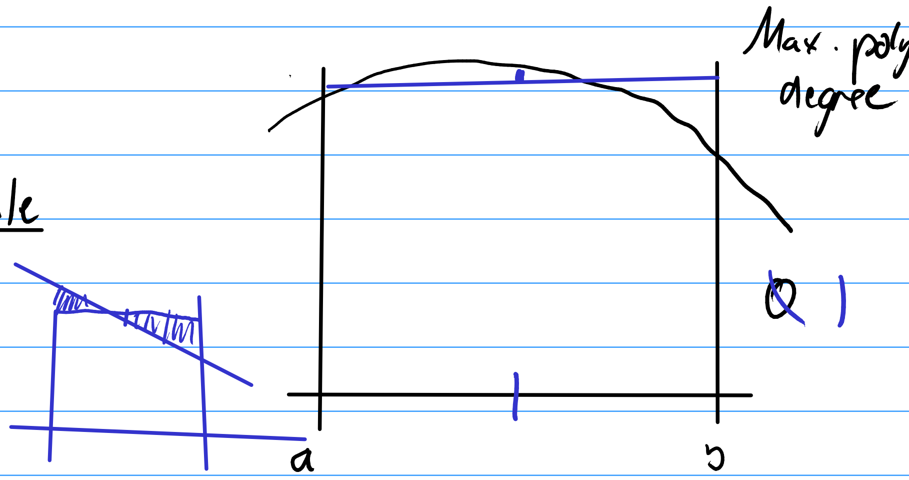
Method of undetermined coefficients

$$b-a = \int_a^b 1\,dx = w_1\cdot 1 + w_2\cdot 1 + \cdots + w_n\cdot 1$$

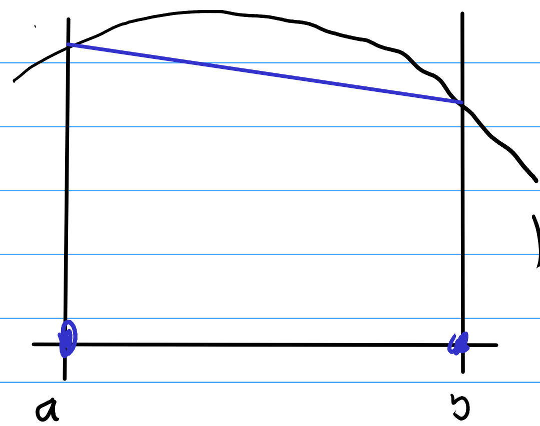
$$\frac{1}{\cancel{2}\;k+1}\left(b^{\cancel{2}\,k+1} - a^{\cancel{2}\,k+1}\right) = \int_a^b x^k\,dx = w_1\cdot x_1^k + w_2\cdot x_2^k + \cdots + w_n\cdot x_n^k$$

Examples

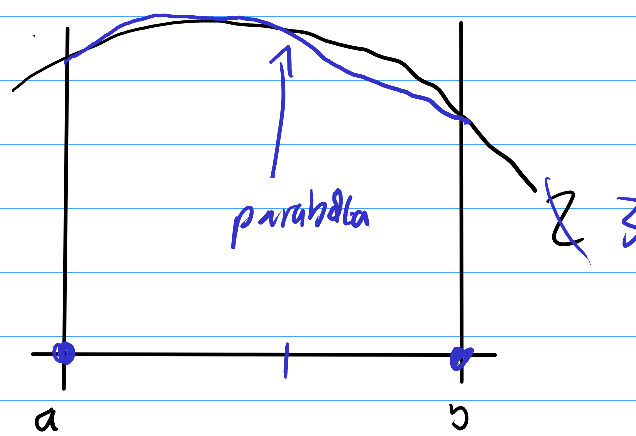
Midpoint rule



Trapezoidal rule



Simpson's rule



Idea: Can estimate error by computing multiple quadratures (such as midpoint + trapz.)

8.2 Accuracy and stability (For Newton-Cotes)

Accuracy: Let p_n be an interpolant of f at nodes $x_1, \dots, x_n$ ← deg. $n-1$.

Recall: $\sum_{i=1}^{n} \omega_i f(x_i) = \int_a^b p_n(x)$

$$\left| \int_a^b f(x)\,dx - \int_a^b p_n(x)\,dx \right|$$

$$\leq \int_a^b | f(x) - p_n(x) |\,dx$$

$$\leq (b-a)\, \| f - p_n \|_\infty$$

intp. err. est.

$$\leq \underbrace{(b-a)}_{n} \frac{h^n}{4} \cdot \| f^{(n)} \|_\infty$$

$$\leq \frac{h^{n+1}}{4} \| f^{(n)} \|_\infty$$

Stability: $\hat{f}(x) = f(x) + e(x)$

$$\left| \sum_i w_i f(x_i) - \sum_i w_i \hat{f}(x_i) \right|$$

$$= \left| \sum_i w_i e(x_i) \right|$$

$$\leq \sum_i |w_i| \underbrace{|e(x_i)|}_{\leq \|e\|_\infty}$$

$$\leq \left(\sum_i |w_i| \right) \cdot \|e\|_\infty$$

Oscillatory weights $\rightsquigarrow$ bad quad. rule

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