$$\int_0^1 \underbrace{\int_0^1 f(x,y)\,dx}_{}\,dy = \int_0^1 \underbrace{\sum_i w_i f(x_i,y)}_{}\,dy$$

$$= \sum_j w_j \sum_i w_i f(x_i,x_j)$$

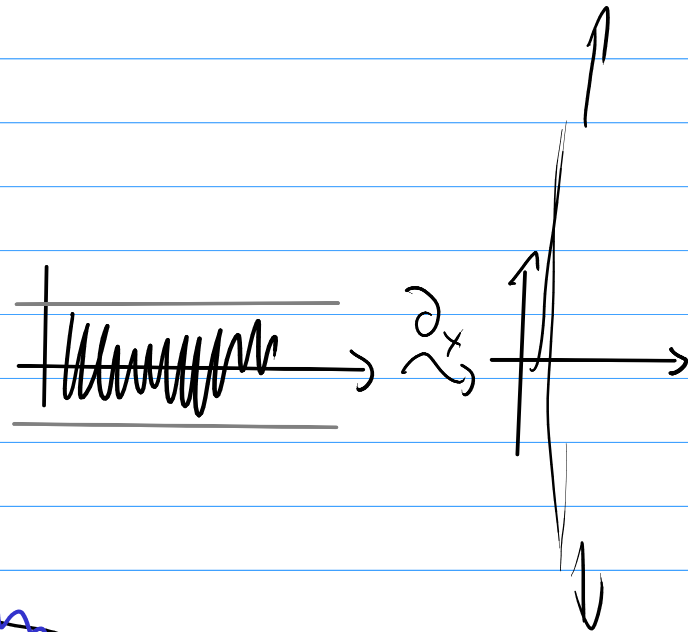
$$= \sum_{i,j} w_i \cdot w_j \; f(x_i,x_j)$$

"tensor product" (quadrature)

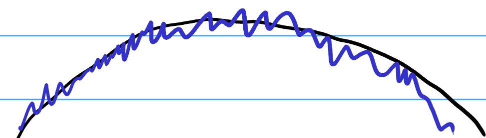
8.5 Numerical Differentiation

Just don't.

- Cancellation
- bounded? NO



- noise



- less accurate

For polynomial interpolant of deg n:

- interpolation h^{n+1}
- quadrature h^{n+2}
- differentiation h^{n}

How?

- Compute interp. coefficients, sum against ∂_x(basis)
- Finite differences

Finite differences

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$ ("forward difference")

(→ bw diff.)

Taylor: $$f(x+h) = f(x) + f'(x)h + f''(x) \cdot \frac{h^2}{2} + \dots$$

$(h \to 0)$ $$= f(x) + f'(x)h + O(h^2)$$

$$\frac{f(x+h) - f(x)}{h}$$

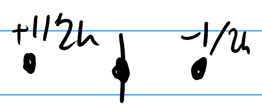
$$= \frac{\cancel{f(x)} + f'(x)h + O(h^2) - \cancel{f(x)}}{h}$$

$$= f'(x) + \frac{O(h^2)}{h} = f'(x) + O(h)$$

→ first order accurate

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$ ("centered difference")

→ second order



8.6 Richardson extrapolation

Suppose we have pth order accurate approx. $\hat{F}$ to some F.

$$F = \hat{F} + O(h^p)$$

Pull out one more term in error expansion:

$$F = \hat{F} + a h^p + O(h^q) \qquad q > p$$

usually: $q = p+1$

Suppose you've applied $\hat{F}$ with h_1, h_2

Find some numbers: α, β

$$\alpha \hat{F}(h_1) + \beta \hat{F}(h_2)$$

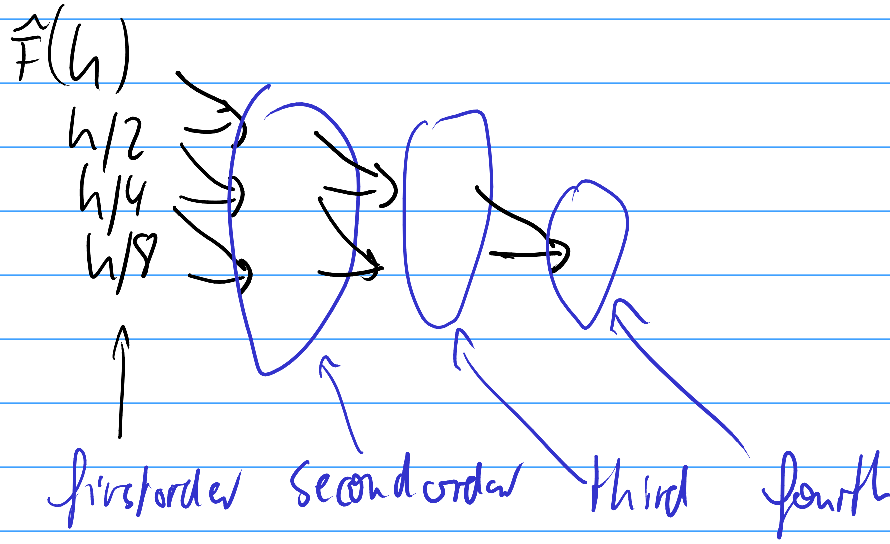
to cancel h^p error term.

$$\alpha a \cdot h_1^p + \beta a \cdot h_2^p = 0 \quad | : a$$

$$\alpha + \beta = 1$$

$$\alpha = \frac{h_2^p}{h_2^p - h_1^p} \qquad \beta = 1 - \alpha$$

WS28p1



This for quadrature : Romberg integration

9 Initial Value Problem

✓ linear systems

✓ nonlinear systems

✗ systems w/derivatives

ODE (d. in one dir)

IVP 9

BVP 10

PDE (d. in multiple dir) 11