9 IVPs

$$y : [0, T] \longrightarrow \mathbb{R}^n$$

$$\left.\begin{array}{l} f(t, y, y', y'', \ldots, y^{(k)}) = 0 \\ y^{(k)}(t) = f(t, y, y', y'', \ldots y^{(k-1)}) \end{array}\right\} \begin{array}{l} \text{implicit} \\ \text{explicit} \end{array}$$

kth order ODES

ordinary differential equations

$$y'(t) = \alpha y(t)$$

$$\text{initial values} \left\{ \begin{array}{l} y(0) = g_0 \\ y'(0) = g_1 \\ y''(0) = g_2 \\ \vdots \\ y^{(k-1)}(0) = g_{k-1} \end{array} \right.$$

imposing cond. at other end $t = T$ gives a BVP.

ODE + IV = IVP

$$y''(t) = f(y(t))$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}'(t) = \begin{pmatrix} z_2(t) \\ f(z_1(t)) \end{pmatrix}$$

$$z_1''(t) = (z_1')'(t) = z_2'(t) = f(z_1(t))$$

Properties

autonomous f does not depend on t

$$y'(t) = f(t, y(t))$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}'(t) = \begin{pmatrix} 1 \\ f(z_1(t) z_2(t)) \end{pmatrix}$$

$$z_1(t_0) = t_0$$

linear $f(t, y) = A(t)\, y + b$

lin. + homogeneous like this but with $b = 0$

$$f(t, y) = A(t) \cdot y$$

constant-coefficient $A(t) = A$

9.1 Existence, Uniqueness, Cond.

Consider:

$$\circledast \begin{cases} y'(t) = f(y) \\ y(t_0) = y_0 \end{cases} \qquad \begin{aligned} \hat{y}'(t) &= f(\hat{y}) \\ \hat{y}(t_0) &= \hat{y}_0 \end{aligned}$$

Picard-Lindelöf's theorem:

If f Lipschitz continuous ("bounded slope")

i.e. if $\| f(x) - f(\hat{x}) \| \leq L \; \| x - \hat{x} \|$

Lipschitz constant.

then:

- there exists a solution of $\circledast$ in a neighborhood of t_0
- $\| \hat{y}(t) - y(t) \| \leq e^{L(t-t_0)} \; \| \hat{y}_0 - y_0 \|$

$\rightarrow$ solution unique

Gronwall's lemma

<u>Conditioning</u> (in numerical ODE-speak

both conditioning of IVP → "stability"

and stability of method → "stability"

ODE <u>stable</u> iff solution depends continuously on the initial cond.

For all $\varepsilon > 0$
there exists a $\delta > 0$

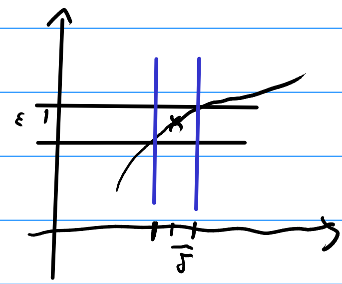
such that

$\| y - \hat{y}_0 \| < \delta$
$\Rightarrow$
$\| y(t) - \hat{y}(t) \| < \varepsilon$ for all $t \geq t_0$.

f continuous at x_0:

for all $\varepsilon > 0$
there exists a $\delta > 0$

$\| x - x_0 \| < \delta$
$\Rightarrow$
$\| f(x) - f(x_0) \| < \varepsilon$



ODE <u>asymptotically stable</u> if $\| \hat{y}(t) - y(t) \| \to 0$.

Example I $\quad y'(t) = \lambda y(t) \qquad \lambda = a+ib \quad a,b \in \mathbb{R}$

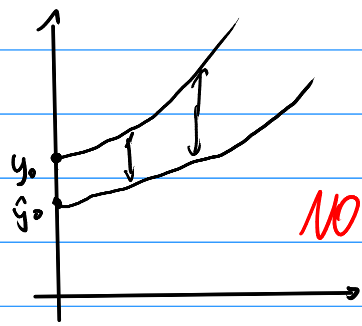
$y(0) = y_0 \qquad \lambda \in \mathbb{C}$

$$y(t) = y_0 e^{\lambda t} = y_0 e^{(a+ib)t} = y_0 e^{at} e^{ibt}$$

$|\cos(bt) + i\sin(bt)| = 1$

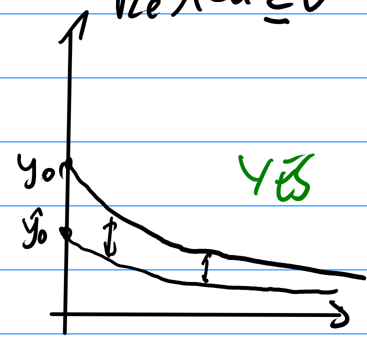
Claim: stable if $\operatorname{Re}\lambda = a \le 0$

$a = \operatorname{Re}\lambda > 0$



NO - not stable

$\operatorname{Re}\lambda = a \le 0$



YES

also asymptotically stable

Example II $\quad \vec{y}'(t) = A\vec{y}(t)$

$\vec{y}(0) = \vec{y}_0$

Assume A diagonalizable; $V^{-1}AV = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\vec{w} = V^{-1}\vec{y}$

new IVP
$$\begin{cases} \vec{w}'(t) = V^{-1}\vec{y}'(t) = V^{-1}A\vec{y}(t) = V^{-1}AV\vec{w}(t) \\ \qquad = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\vec{w}(t) \\ \vec{w}'(0) = V^{-1}\vec{y}(0) \end{cases}$$

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$\rightsquigarrow \vec{y}(t) = V\vec{w}$. $\qquad$ Stable iff $\operatorname{Re}(\lambda_1) \dots \operatorname{Re}(\lambda_n) \le 0$.