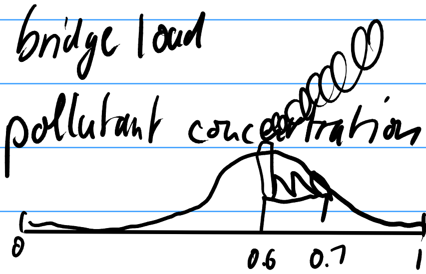
time step

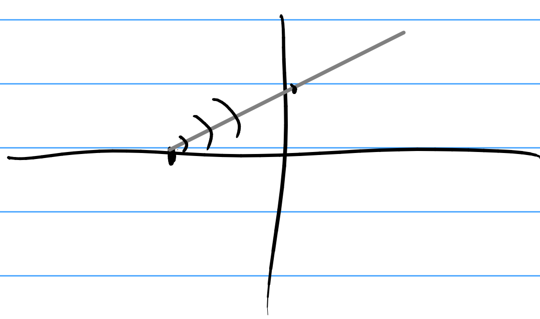
$\downarrow$

$y_{k+1} = y_k + \dots$

| ODE/IVP | BVP: |
|---|---|
| • population dynamics | • bridge load |
| $y_1 = -\alpha y_2$ <br> $y_2 = \beta y_2$ | • pollutant concentration |
| • chemical reactions |   |
| • equations of motion | • temperature <br>  |

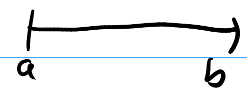
$y'(t) = A\, y(t)$

$y'(t) = \lambda\, y(t)$



## (10) BVP

$$u''(x) + p(x)u'(x) + q(x)\, u(x) = r(x)$$




with **Dirichlet boundary cond.** $u(a) = u_a$ $\qquad$ $u(b) = u_b$

– or –

**Neumann BC** $u'(a) = v_a$ $\qquad$ $u'(b) = v_b$

– or –

**Robin BC** $\alpha_a u(a) + \beta u'(a) = w_a$

### Fully general version:

ODE: $\vec{y}'(x) = f(\vec{y}(x))$ $\qquad$ $y: \mathbb{R}^{[a,b]} \to \mathbb{R}^n$

BC: $\vec{g}(y(a), y(b)) = \vec{0}$ $\qquad$ $g: \mathbb{R}^{2n} \to \mathbb{R}^n$

Example: $B_a\, y(a) + B_b\, y(b) = c$

## 10.1 Existence, Uniqueness, Conditioning.

nonlinear case: ?

Special case: linear

$$\vec{y}'(x) = A(x)\,\vec{y}(x) + b(x)$$

$$B_a\,\vec{y}(a) + B_b\,\vec{y}(b) = c$$

matrix

First, worry about homogeneous

$$(H)\quad \begin{cases} \vec{y}'(x) = A(x)\,\vec{y}(x) + \cancel{b(x)} \\ B_a\,\vec{y}(a) \cancel{+ B_b\,\vec{y}(b)} = c \end{cases}$$

$$\vec{y}_i(a) = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \qquad i = 1 \dots n$$

Matrix $Y(x) = (\vec{y}_1\ \vec{y}_2 \dots \vec{y}_n)$ fundamental solution matrix.

$$Q := B_a \cdot Y(a) + B_b \cdot Y(b)$$

Unique solution to $(H)$ exists iff $Q$ is invertible.

$$\underbrace{\Phi(x) := Y(x)}_{\text{combine columns}}\ Q^{-1} \qquad \text{hom solution}$$

$y_0(x) = \Phi(x)\,c$

Define Green's Function

$$G(x, x') := \begin{cases} \Phi(x)\, B_a \cdot \Phi^{-1}(x') & x' \leq x \\ -\Phi(x)\, B_b \cdot \Phi^{-1}(x') & x' > x \end{cases}$$

scalar $\in [a,b]$

Then

$$y(x) := \Phi(x) c + \int_a^b G(x, x')\, b(x') \, dx'$$

hom. solution — RHS of ODE

Computationally: not optimal.

$$\|\Delta y\|_\infty \leq \underbrace{\max\left(\|\Phi\|_\infty, \|G\|_\infty\right)}_{\text{abs. condition number}} \cdot \left(\|\Delta c\|_1 + \int \|\Delta b\|_1 \, dx\right)$$

→ well-posed