Galerkin / finite element methods

$$\Delta u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \dots\right) = f(x)$$

1D

$$u''(x) = f(x) \qquad u(a) = 0 = u(b)$$

Step I: Pick basis

unknowns

$$w(x) = \sum_{i=1}^{n} \alpha_i \, \varphi_i(x)$$

Step II: Find residual

$$0 \overset{!}{=} r(x) = w''(x) - f(x) = \sum_i \alpha_i \varphi_i''(x) - f(x)$$

not going to happen

Step III: Write down some equations

Pick n

Idea: $(r, \psi_i) = \int_a^b r(x)\, \psi_i(x)\, dx = 0$

Residual $\perp$ Test functions

Lazy / convenient choice: Test f. = Basis f.

$$(r, \varphi_i) = 0$$

"Galerkin method"

$$0 = (r, \varphi_i) = \int w''(x)\,\varphi_i(x) - f(x)\,\varphi_i(x)\,dx$$

$$\int w''(x)\,\varphi_i(x)\,dx = \underbrace{\left[w'(x)\,\varphi_i(x) \right]_a^b}_{=0} - \int_a^b w'(x)\,\varphi_i'(x)\,dx$$

$\leftarrow$ if basis obeys BC: 0

System of equations:

$$-\int_a^b w'(x)\,\varphi_i'(x)\,dx \overset{!}{=} +\int f(x)\,\varphi_i(x)\,dx$$

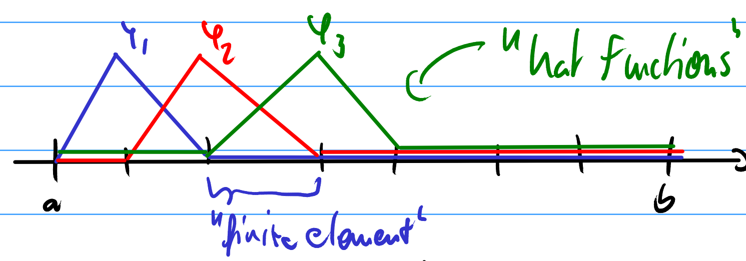
$$-\int_a^b \sum_{j=1}^n \alpha_j\,\varphi_j'\,\varphi_i'(x)\,dx = \dots$$

$$\sum_{j=1}^n \alpha_j \underbrace{\int_a^b \varphi_j'(x)\,\varphi_i'(x)\,dx}_{S_{ij}\ \text{"stiffness matrix"}} = \dots$$

$$(S\vec{\alpha})_i \qquad - \int f(x)\,\varphi_i(x)\,dx$$

Care needed w/ global bases because S possibly dense.

<u>Pick basis:</u> "finite element basis" $\qquad \varphi_i$ has kink vs. derivative $\rightarrow$ Sobolev space



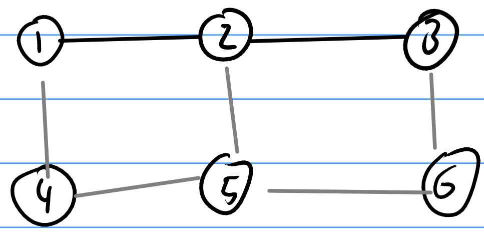
"piecewise linear Lagrange basis"

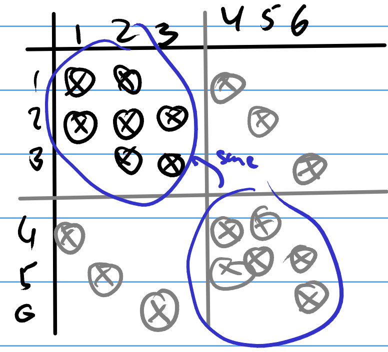
11 Previews

CS 555 → PDE

CS 556 → sparse linear algebra / MG

11. part 1 — Sparse linear algebra

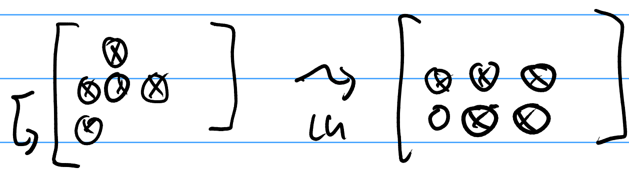




Terminology: graph, neighbor, edge

Problem: LU can generate non-zeros (tons of them)

"Sparse fill-in"



Idea: Renumber so that non-zeros end up close to diagonal.

"start with one node examine neighbors, number low-degree neighbors first, repeat"

"Cuthill-McKee" / "minimum degree"

closely related to breadth-first search ("BFS")

Reverse Cuthill-McKee = RCM