

Sparse direct: upsides/downsides

⊖ scalability (not applicable)

⊕ robust
⊖ storage cost.

11.1 Stationary iterative methods

Idea: $A = M - N$ to solve $Ax = b$
↑ actually only invert this

$$Mx_{k+1} = Nx_k + b$$

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b$$

why? make sol. x^* a fixed point

Fixed point iteration

converges if $\rho(M^{-1}N) < 1$

Attempt 1: $M = \text{diag}(A)$ Jacobi method

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h))}{h^2}$$

$$A = D + L + U$$

$$M = D \quad N = -(L + U)$$

Jacobi converges if
 A is diagonally
dominant.

$$|A_{ii}| > \sum_{i \neq j} |A_{ij}|$$

$$(x_{k+1})_i = \frac{b_i - \sum (\text{offdiag}) \cdot x_k}{a_{ii}}$$

Gauss-Seidel: $M = D+L$ $N = -U$

SOR: $M = \frac{1}{\omega} D + L$ $N = \left(\frac{1}{\omega} - 1\right) D - U$

direction amplification factor: ω

(11.2) Conjugate gradient method

Assume: A SPD

Goal: Minimize: $\phi(x) = \frac{1}{2}x^T A x - x^T b$

solution of $Ax^* = b$
 $\phi(x^*) = 0$

Idea:

$x_0 =$ (starting guess)

$$x_{k+1} = x_k + \alpha_k s_k$$

What's α ?

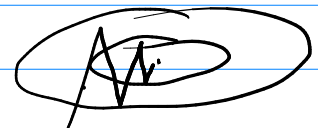
$$0 \stackrel{!}{=} \frac{\partial}{\partial \alpha} \phi(x_k + \alpha s_k) = 0 \quad \begin{array}{l} \text{(residual)} \\ \downarrow \end{array} \quad \nabla \phi(x) = Ax - b$$
$$= \nabla \phi(x_{k+1}) \cdot s_k = -r_{k+1} \cdot s_k$$

Need: residual \perp search direction

What's s_k ?

$$s_k = r = -\nabla \phi(x_k)$$

\uparrow
steepest descent



Idea: s_k, s_l are A -orthogonal (except $k=l$)

Terminology: x, y are A -orthogonal

$$\text{iff } x^T A y = 0$$

$$e_0 = x_0 - x^* = \sum_i \delta_i s_i \quad \textcircled{\otimes}$$

$$s_k^T A e_0 = \sum_{i=1}^n \delta_i s_k^T A s_i = \delta_k s_k^T A s_k$$

$$\delta_k = \frac{s_k^T A e_0}{s_k^T A s_k} = \frac{s_k^T A (e_0 + \sum_{i=1}^{k-1} \alpha_i s_i)}{s_k^T A s_k} = \frac{s_k^T A e_k}{s_k^T A s_k} = -\alpha_k$$