Sparse direct: upsides/downsides

$\ominus$ scalability (not applicable)

$\oplus$ robust

$\ominus$ storage cost.

(11.1) Stationary iterative methods

Idea: $A = M - N$ to solve $Ax = b$

actually only invert this

$$M x_{k+1} = N x_k + b$$

why? make sol. x^* a fixed point

$$x_{k+1} = M^{-1} N x_k + M^{-1} b$$

Fixed point iteration

converges if $\rho(M^{-1}N) < 1$

Attempt 1: $M = \mathrm{diag}(A)$ Jacobi method

Jacobi converges if A is diagonally dominant.
$$|A_{ii}| > \sum_{i \neq j} |A_{ij}|$$

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$A = D + L + U$$

$$M = D \qquad N = -(L+U)$$

$$(x_{k+1})_i = \frac{b_i - \Sigma(\text{offdiag}) \cdot x_k}{a_{ii}}$$

<u>Gauss-Seidel:</u> $M = D + L \qquad N = -U$

<u>SOR:</u> $M = \frac{1}{\omega} D + L \qquad N = \left(\frac{1}{\omega} - 1\right) D - U$

direction amplification factor: ω

(11.2) Conjugate gradient method

Assume: A SPD

Goal: Minimize: $\varphi(x) = \frac{1}{2}x^T A x - x^T b$

solution of $Ax^* = b$

$\varphi(x^*) = 0$

Idea:

$x_0 = \langle \text{starting guess} \rangle$

$x_{k+1} = x_k + \alpha_k s_k$

What's α?

$$0 \overset{!}{=} \frac{\partial}{\partial \alpha} \varphi(x_k + \alpha_k s_k) = 0 \qquad \nabla \phi(x) = \overset{\substack{(\text{residual}) \\ -r \\ \parallel}}{Ax - b}$$

$$= \nabla \varphi(x_{k+1}) \cdot s_k = -r_{k+1} \cdot s_k$$

Need: residual $\perp$ search direction

What's s_k?

$$s_{\cancel{k}} = r = -\nabla \phi(x_k)$$

Steepest descent

<u>Idea:</u> s_k, s_ℓ are A-orthogonal (except $k=\ell$)

<u>Terminology:</u> x, y are <u>A-orthogonal</u>

iff $x^T A y = 0$

$$e_0 = x_0 - x^* = \sum_i \delta_i s_i \quad ⊗$$

$$s_k^T A e_0 = \sum_{i=1}^{n} \delta_i \, s_k^T A s_i = \delta_k s_k^T A s_k$$

$$\delta_k = \frac{s_k^T A e_0}{s_k^T A s_k} = \frac{s_k^T A \left(e_0 + \sum_{i=1}^{k-1} \alpha_i s_i\right)}{s_k^T A s_k} = \frac{s_k^T A e_k}{s_k^T A s_k} = -\alpha_k$$