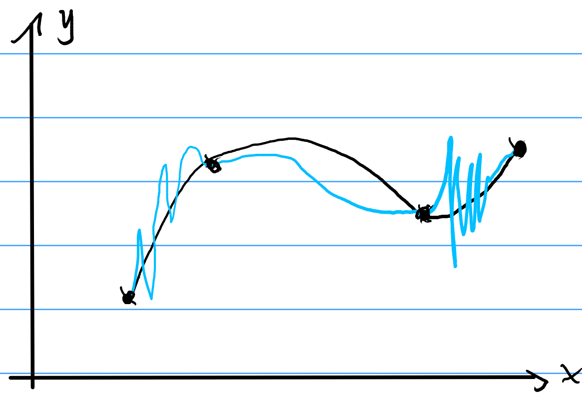
7 Interpolation



given

x	y
x_1	y_1
x_2	y_2
x_m	y_m

0 [Goal: Put a function through here.
Impossible! (or: really non-unique)

Seeking: $\alpha_1, \dots, \alpha_m$] 2

in

linear interpolation

1 [$f(x) = \sum_{j=1}^{m} \alpha_j \varphi_j(x)$ s.t. $f(x_i) = y_i$ $(i=1 \dots m)$

$$\downarrow$$

$$f(x_i) = \sum_{j=1}^{m} \underbrace{\varphi_j(x_i)}_{V_{ij}} \alpha_j = (V\vec{\alpha})_i$$

generalized Vandermonde matrix

book: "basis matrix"

Interpolation: $V\vec{\alpha} = \vec{y}$

$$V \begin{pmatrix} \text{basis} \\ \text{coeffs} \end{pmatrix} = \begin{pmatrix} \text{node} \\ \text{values} \end{pmatrix}$$

<u>Why? What's the point?</u> $\rightarrow$ Enable us to work with the function

- Evaluate "anywhere"
- Differentiate
- Integrate

$f'(x) = \sum_j \alpha_j \varphi_j'(x)$

$\hookrightarrow V_{\varphi'} V_{\varphi}^{-1} \vec{y}$

<lec 26>

Office hours moved to 11am

<u>Existence:</u> $\vec{y} \in$ column-span V

<u>Uniqueness:</u> $\mathcal{N}(V) = \{0\}$

<u>Conditioning:</u> $\mathrm{cond}(V)$

<u>Questions:</u>
- basis?
- nodes?

Node placer demo

7.1 Nodes and bases

Monomial basis: $1, x, x^2, \ldots, x^{n-1}$ ask

V for monomials: (un-generalized) Vandermonde matrix

Demo

Lagrange basis: Idea: Make $V = I$

In general $\varphi_j(x_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta_{ij}$

nodes x_1, x_2, x_3 Want: $\varphi_1(x_1) = 1$ $\varphi_1(x_2) = 0$ $\varphi_1(x_3) = 0$

$$
\begin{cases}
\varphi_1(x) = \dfrac{\overbrace{(x-x_2)(x-x_3)}^{\text{① make it zero}}}{\underbrace{(x_1-x_2)(x_1-x_3)}_{\text{② make it one at } x_1}} \\[2ex]
\varphi_2(x) = \dfrac{(x-x_1)}{(x_2-x_1)} \dfrac{(x-x_3)}{(x_2-x_3)} \\[2ex]
\varphi_3(x) = \dfrac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}
\end{cases}
$$

$$\varphi_j(x) = \frac{\prod_{k=1, k\neq j}^{m} (x - x_k)}{\prod_{k=1, k\neq j}^{m} (x_j - x_k)}$$

Demo

Q [Downsides? → expensive to evaluate; → differentiate that!

WS24p1

⚠ Changed

Newton basis

$$\varphi_j(x) = \prod_{k=1}^{j-1} (x - x_k)$$

$V_{ij} = \varphi_j(x_i) = 0$ if $i<j$

→ triangular

o [Slick procedure for finding coefficients:

"Divided differences" → hw5

o [What other type of V would be well-conditioned and easy to invert?

Orthogonal (polynomial) bases

o [When are two functions "orthogonal"?

Define an inner/"dot" product on functions

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, \underbrace{w(x)}_{\text{weight}}\, dx$$

Assume $w(x) = 1$ for now.

Can then Gram-Schmidt $1, x, x^2, x^3, \ldots$

→ "Legendre polynomials"

Demo

o [So computing them is annoying. Or is it?

Three-term recurrence (for Legendre)

$$(k+1)P_{k+1}(x) = (2k+1)\,x\,P_k(x) - k\,P_{k-1}(x)$$

→ DLMF

O [Know the first two. So: easy to compute!

<lec 27>

Other bases:

- Chebyshev (on $[-1,1]$ w. weight $1/\sqrt{1-x^2}$
- Hermite (on $(-\infty,\infty)$ w. weight e^{-x^2})

⋮

Chebyshev: $T_k(x) = \cos(k \cos^{-1}(x))$ → hw 4

- actually a polynomial
- values between -1 and 1 uniformly
  → uniform dist of error

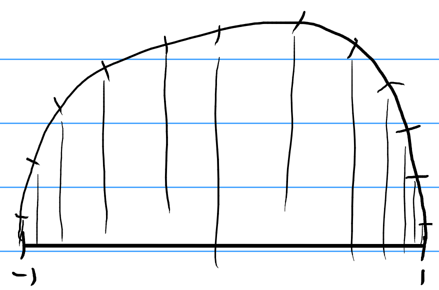
Demo pt. 1

- roots generate a useful set of interpolation nodes
  → minimal max error over whole interval

WS25p1

What are those roots?

$$\cos(k \cos^{-1}(x)) = 0$$
$$\Leftrightarrow \quad k \cos^{-1}(x) = \frac{2i+1}{2}\pi$$
$$\Leftrightarrow \quad \cos^{-1}(x) = \frac{2i+1}{2k}\pi$$
$$\Leftrightarrow \quad x_i = \cos\left(\frac{2i+1}{2k}\pi\right)$$



o bunch together near interval ends, confirming earlier demo experience

Demo pt 2

7.2 Error result

Assume $x_1 < \cdots < x_n$.

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!} \overbrace{(x-x_1)(x-x_2)\cdots(x-x_n)}^{\otimes}$$

ξ — depends on x, unknown ②

$(x-x_1)(x-x_2)\cdots(x-x_n)$ — Error zero at nodes ①

(link to proof posted)

If we have $|f^{(n)}([x_1, x_n])| \leq M$:

$$\max_x |(f - p_{n-1})(x)| \leq \frac{Mh^n}{4n}$$

where $h = \max_i |x_{i+1} - x_i|$.

Why does Chebyshev-like "bunching" work?

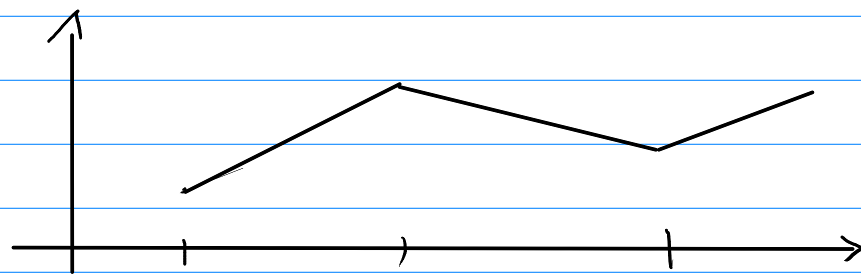
Because $\otimes$ is small near interval ends!

7.3 Piecewise polynomial interpolation

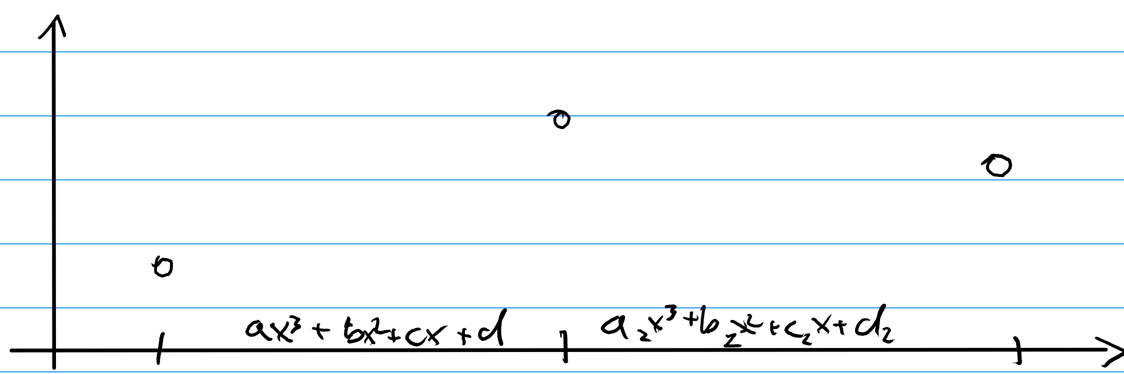
High-degree polynomial interpolation can be expensive (n^3) as shown. Possible to make cheaper, but: need more high-tech.

Low-tech alternative: piecewise interp.

piecewise linear



piecewise cubic



count # parameters, # equations

- derivatives available? use them → "Hermite" interp
- enforce two continuous derivatives → cubic "splines"

OK now? What about bdry points?
"Natural": $p''(\text{ends}) \stackrel{!}{=} 0$