✓ linear systems

✓ non linear systems

✗ systems w/derivatives

ODEs (d. in one direction) PDEs (d. in multiple dir.) 11

IVP 9 BVP 10

Applications:

ODE/IVP

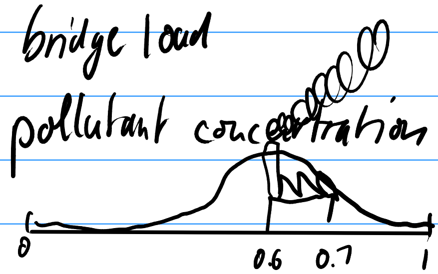
- population dynamics

$$y_1 = -\alpha y_2 \leftarrow \text{prey}$$
$$y_2 = \beta y_2 \leftarrow \text{predator}$$

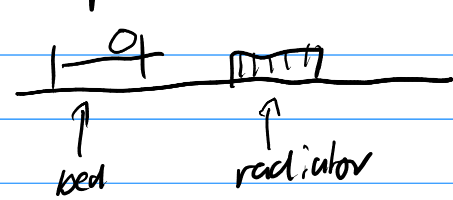
- chemical reactions
- equations of motion

BVP:

- bridge load
- pollutant concentration



- temperature



9 Initial Value Problems

$y : [0,T] \to \mathbb{R}^n$ desired solution

$$f(t, y, y', y'', \dots, y^{(k)}) = 0 \quad \text{(implicit)}$$
$$y^{(k)}(x) = f(t, y, y', \dots, y^{(k-1)}) \quad \text{(explicit)}$$

(kth-order) ODE

solvable as is?

consider simple example: $y' = \alpha y$

→ need initial values. how many?

$y(0) = g_0$
$y'(0) = g_1$
$\vdots$
$y^{(k-1)} = g_{k-1}$

could swap some derivatives for conditions at other end:

$y(T) = \tilde{g}_0$ } → BVP
...

ODE + Initial values = IVP

Can always reduce to first order:

$$y''(t) = f(y) \quad \to \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}'(t) = \begin{pmatrix} y_2(t) \\ f(y_1(t)) \end{pmatrix}$$

$$\leadsto y_1''(t) = (y_1'(t))' = y_2'(t) = f(y_1(t))$$

Properties

autonomous : f does not depend on t

O [can always make autonomous using extra variable: $y_0'(t) = 1$ $y_0(0) = 0$

→ will omit explicit t dependency

linear : $f(x) = A(t)\,x + b$

lin + homogeneous : $f(x) = A(t)\,x$

constant - coefficient : $A(t) = A$

9.1 Existence, Uniqueness, Conditioning

Consider:

$$\otimes \begin{cases} y'(t) = f(y) \\ y(t_0) = y_0 \end{cases} \qquad \begin{matrix} \hat{y}(t) = f(\hat{y}) \\ \hat{y}(t_0) = \hat{y}_0 \end{matrix}$$

Assume: f Lipschitz continuous ("bounded slope")

$$\| f(y) - f(\hat{y}) \| \leq L \| y - \hat{y} \|$$

Lipschitz constant

If so: ("Picard-Lindelöf theorem")

- there exists a solution of $\otimes$ in a neighborhood of t_0
- $\| \hat{y}(t) - y(t) \| \leq e^{L(t-t_0)} \| \hat{y}_0 - y_0 \|$

[What does this mean for uniqueness?

Conditioning: (in ODE-speak: "stability")

for us: "...of a method"
for them: used for both ODEs and methods

ODE stable iff solution continuously dependent on on initial condition

in ε-δ speak:

For all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$\|\hat{y}_0 - y_0\| \le \delta \Rightarrow \|\hat{y}(t) - y(t)\| \le \varepsilon$ for all $t \ge t_0$.

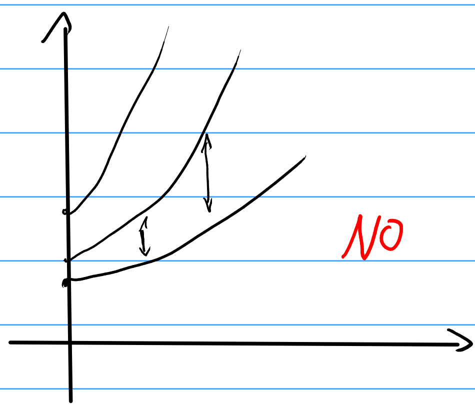
ODE asymptotically stable iff $\|\hat{y}(t) - y(t)\| \to 0 \quad (t \to \infty)$

Example I $\quad y'(t) = \lambda y \qquad \lambda = a + ib$
$\quad y(0) = y_0$

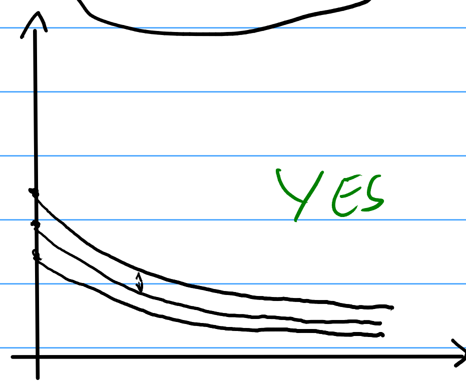
Solution: $y(t) = y_0 e^{\lambda t} = y_0 e^{at} e^{ib} \quad \rightarrow |y(t)| = |y_0 e^{at}|$

0 [Stable?

$a = \mathrm{Re}\,\lambda > 0$



$a = \mathrm{Re}\,\lambda \leq 0$



Example II $\quad \vec{y}'(t) = A\vec{y}(t)$
$\quad \vec{y}(0) = \vec{y}_0$

Assume A is diagonalizable $\quad V^{-1}AV = \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{D}$

$\vec{w} = V^{-1}\vec{y}$

newIVP
$$\begin{cases} \vec{w}'(t) = V^{-1}\vec{y}'(t) = V A \vec{y}(t) = V A V^{-1} \vec{w}(t) \\ \qquad = D\,\vec{w}(t) \\ \\ \vec{w}_0 = V^{-1}\vec{y}_0 \end{cases}$$

$\rightsquigarrow \vec{y}(t) = V\vec{w}(t)$

0 [Stable when?

WS29p1

<lec32>

- exam on Wed.
- finish WS29

9.2 Numerical Methods (pt. I)

Discrete times: $t_1, t_2, \ldots$ $\quad t_{i+1} = t_i + h$ (for now)

Discrete function values: $y_k \approx y(t_k)$

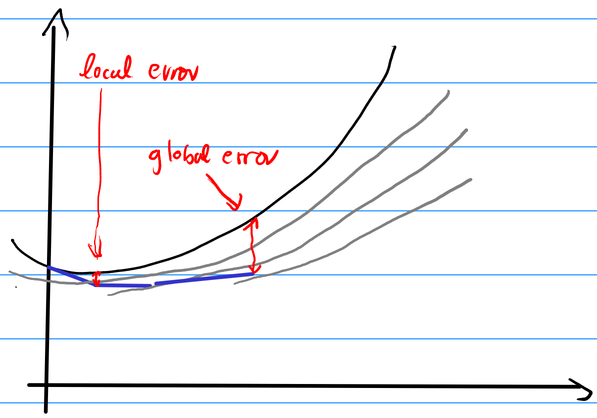
IVP: $y'(t) = f(y(t))$, $y(t_0) = y_0$	$\longleftrightarrow$	**Integral equation:** $(\otimes)\quad y(t) = y_0 + \int_{t_0}^{t} f(y(\tau))\, d\tau$

Idea: Throw quadrature rule at $\otimes$. $\quad$? [Which? (for solvability)

$y_{k+1} = y_k + h f(t_k, y_k)$ (Euler's method)

Demo

9.3 Accuracy and Stability



local error: $\ell_k = y_k - u_{k-1}(t_k)$ (where u_ℓ solves the ODE with IV $u(t_\ell) = y_\ell$)

global error: $y(t_k) - y_k$

- Is global error = Σ (local errors)?
- Interest analogy.
- Local error much easier to estimate!

Time integrator of order p :$\Leftrightarrow$ $\ell_k = O(h^{p+1})$

one higher than one might expect

Local error per length-1 step: $\frac{1}{h} \cdot O(h^{p+1})$ (assuming limited "accrual" of propagated error)

$O(h^p)$

Stability of a method

Instability can be caused:

- by ODE — [Will not really distinguish the two.
- by the method

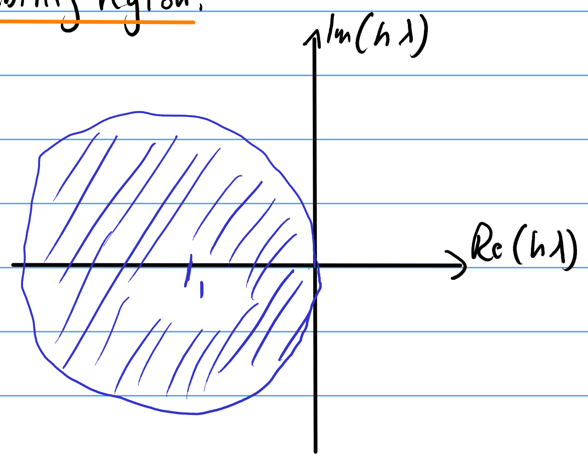
in Forward Euler, for $y'(t) = \lambda y(t)$:

$$y_k = y_{k-1} + h\lambda y_{k-1}$$
$$= \underbrace{(1+h\lambda)}_{\text{growth factor}} y_{k-1}$$

stable $\Leftrightarrow$ $|1+h\lambda| < 1$

$$y_k = (1+h\lambda)^k y_0$$

"Stability region":



Stability in the nonlinear case

Fw Euler: (A) $y_{k+1} = y_k + h f(y_k)$

Taylor: (B) $y(t_{k+1}) = y(t_k) + h f(y(t_k)) + O(h^2)$

$$(A)-(B) = e_{k+1} = y_{k+1} - y(t_{k+1}) = \underbrace{\underbrace{(y_k - y(t_k))}_{e_k=} + h\big(f(y_k) - f(y(t_k))\big)}_{\text{propagated error}} + \underbrace{\underbrace{O(h^2)}_{\ell_k:=}}_{\text{local error}}$$

$$\bar{J}_f := \int_0^1 J_f(\alpha y_k + (1-\alpha) y(t_k))\, d\alpha$$

$$e_{k+1} = e_k (I + h \bar{J}_f) + \ell_k$$

Errors do not grow if $\rho(I + h\bar{J}_f) \leq 1$.

Difference from linear, scalar result?

Need intermediate result: a (not "the") mean value theorem

LEFT OUT

Assume f differentiable:

$$\int_0^1 J_f(\underbrace{\alpha y + (1-\alpha) x}_{\otimes})\, d\alpha\, (y - x)$$

$$= \int_0^1 \frac{\partial}{\partial \alpha}(f(\otimes))\, d\alpha$$

$$= [f(\otimes)]_0^1 = f(y) - f(x)$$

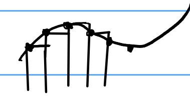
"Original" MVT: $\exists c: f'(c) = \frac{f(b)-f(a)}{b-a}$

(⚠ Bug in book)

Then: $$f(y_k) - f(y(t_k)) = \underbrace{\int_0^1 J_f(\alpha y_k + (1-\alpha) y(t_k))\, d\alpha}_{\bar{J}_f :=} \cdot \underbrace{(y_k - y(t_k))}_{e_k =}$$

9.4 Numerical Methods (pt. II)

Obtained Euler method by throwing left rectangle rule at:



$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau))\, d\tau$$

$$y_{k+1} = y_k + h\, f(y_k)$$

What if we had used right rectangle rule?

$$y_{k+1} = y_k + h\, f(y_{k+1})$$

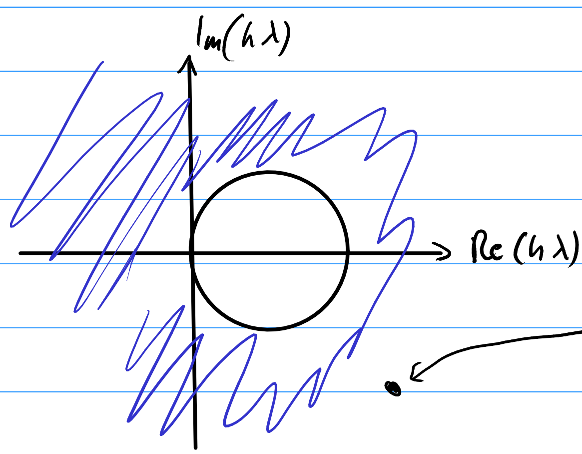
Backward Euler method (earlier Euler method: Forward Euler by contrast)

→ need to solve equation → "implicit method"

(not implicit: "explicit method")

Example: $y'(t) = \lambda y(t)$

$$y_{k+1} = y_k + h\lambda\, y_{k+1} \quad \rightsquigarrow \quad (1 - h\lambda)\, y_{k+1} = y_k$$



$$y_{k+1} = \underbrace{\frac{1}{1-h\lambda}}_{\text{amplification factor}} y_k$$

stable here?!

(yes, but not accurate)

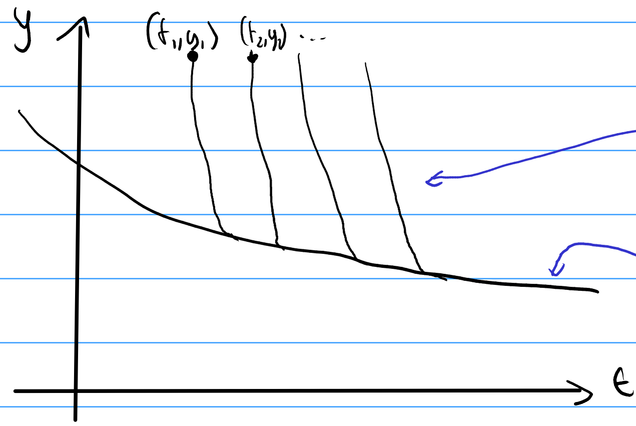
Generically (not just for implicit):

Two restrictions on time step:

- accuracy
- stability

5.5 Stiffness

Demo



Two time scales:

- fast
- slow

$y'(t) = f(y(t))$ stiff if J_f has eigenvalues of very different magnitude

o Why not just "small" or "large"?

o What's the problem with applying explicit methods to stiff problems?

o Express this as conflict between stability and accuracy

o Can an implicit method take arbitrarily large time steps?

Example: acoustic vs. flow

WS 30 p1

(9.6) Numerical Methods (pt. II)

Predictor-corrector methods

Idea: Obtain intermediate result, improve it (with same or different method). E.g.:

- predict with fw Euler $\tilde{y}_{n+1} = y_n + h f(y_n)$
- correct with trapezoidal $y_{n+1} = y_n + \frac{h}{2}\left(f(y_n) + f(\tilde{y}_{n+1})\right)$

↑ Heun's method (2nd order accurate)

More general than P-C: Methods that evaluate f multiple times per step:

- single-step
- multi-stage
- Runge-Kutta

} methods

o [Wikipedia Runge-Kutta:

- point out RK4
- equivalence to Simpson's rule (if no y dependency)

]

o [point out "stages"

o [implicit variants also exist

<4/24>

Systematic view of Runge-Kutta methods: **Butcher tableau**

$$
(1)\quad \begin{cases} r_1 = f(t_k + c_1 h,\ y_k + a_{11}\cdot r_1 + \dots + a_{1s}\cdot r_s) \\ \quad\vdots \\ r_s = f(t_k + c_s,\ y_k + a_{s1}\cdot r_1 + \dots + a_{ss}\cdot r_s) \end{cases}
$$

$$
\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}
$$

$$
(2)\quad y_{k+1} = y_k + b_1\cdot r_1 + \dots + b_s\cdot r_s
$$

(s = number of "stages")

[When is this explicit? implicit? "diagonally implicit?"

Embedded Runge-Kutta method

$y_{k+1} = \dots$

$\bar{y}_{k+1} = \dots$

$$
\begin{array}{c|ccc} c_{\dots} & \cdots & a_{\dots} & \cdots \\ \hline & b_1 & \cdots & b_s \\ & \bar{b}_1 & \cdots & \bar{b}_s \end{array}
$$

estimates of different order, to be used for error estimation / adaptation of h

Recap: Heun's method

$\tilde{y}_{n+1} = y_k + h f(y_n)$

$y_{n+1} = y_k + \frac{h}{2}\left(f(y_n) + f(\tilde{y}_{k+1})\right)$

RK scheme:

$r_1 = f(t_k, y_k)$

$r_2 = f(t_k + h, y_k + h r_1)$

$$
\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}
$$

[Wikipedia

Another Idea: Instead of computing stage values, use history

$$y_{k+1} = \sum_{i=1}^{M} \alpha_i y_{k+1-i} + h \sum_{i=1}^{N} \beta_i f(y_{k+1-i})$$

Known as:

- single-stage
- multi-step
- Adams-Bashforth (if $M=1$)
- Backward Differencing Formulas (BDF) (if $N=1$)

- also exist in implicit variants
- What if there is no history?

Demo: Stability regions

WS31