10 BVPs

Typical: second-order linear ODE

$$u''(x) + p(x)\, u'(x) + q(x)\, u(x) = r(x)$$ → project

Note: x, not t.

with Dirichlet / Neumann / Robin boundary conditions (BCs)

$$\left.\begin{array}{c} u(a) = u_a \\ \text{— or —} \\ u'(a) = v_a \\ \text{— or —} \\ \alpha u(a) + \beta u'(a) = w_a \end{array}\right\} \text{ and } \left\{\begin{array}{c} u(b) = u_b \\ \vdots \end{array}\right.$$

Fully general version:

ODE: $y'(x) = f(y(x))$ $\qquad f: \mathbb{R}^n \to \mathbb{R}^n$

BCs: $g(y(a), y(b)) = 0$ $\qquad g: \mathbb{R}^{2n} \to \mathbb{R}^n$

!

Recall: rewrite higher order ODE as first order system

Does a first-order, scalar BVP make sense?

Example: Linear BCs

$$B_a y(a) + B_b y(b) = c$$

Q: Is this Dirichlet / Neumann / ... ?

A: Can't tell without looking at B_a, B_b.

10.1 Existence, Uniqueness, Conditioning

General, nonlinear case: Harder than root finding

0 [Couldn't say much there either.

Special-case to linear BVP:

$$\circledast \begin{cases} y'(x) = A(x)\,y(x) + b(x) \\ B_a\,y(a) + B_b\,y(b) = c \end{cases}$$

To solve, consider homogeneous IVPs:

$$y_i'(x) = A(x)\,y_i(x)$$

with initial conditions $y_i(a) = e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$

Build fundamental solution matrix $Y(x) = (y_1\, y_2 \dots y_n)$

Let $Q := B_a Y(a) + B_b Y(b)$.

Then $\circledast$ has unique solution iff Q is non-singular.

Let $\Phi(x) := Y(x)\,Q^{-1}$

combine columns!

Observe: $\Phi(x)\,c$ solves homogeneous problem ($\circledast$ with $b(x) = 0$).

Define Green's function

$$G(x,y) := \begin{cases} \Phi(x)\, B_a\, \Phi^{-1}(y) & y \leq x \\ -\Phi(x)\, B_b\, \Phi^{-1}(y) & x < y \end{cases}$$

Then

$$(\text{xxx}) \quad y(x) = \Phi(x)\, c + \int_a^b \underbrace{G(x,y)}_{\text{"kernel" of the integral}} \underbrace{b(y)}_{\text{ODE RHS}}\, dy$$

Conditioning: now easy.

① write without Δs
② insert Δs

$$(\text{xx}) \quad \|\Delta y\|_\infty \leq \underbrace{\max\left(\|\Phi\|_\infty, \|G\|_\infty\right)}_{k} \left(\|\Delta c\|_1 + \int \|\Delta b(y)\|_1\, dy \right)$$

function norms　　vector norms

For perturbed problem $(b(y) + \Delta b(y),\ c + \Delta c)$, (xx) gives bound on change in y.

○ [Also: continuous dependence on data: well-posed.

○ [Can verify that (xxx) is solution by plug'n'chug.

Uniqueness is the only unproven bit.

○ [Is this a computational procedure? → project

(10.2) Numerical methods

Idea: can already solve IVPs.

Problem: Don't know **all** left BCs

Shooting method

Demo

[What about systems? (Cannons are aimed in 2D!)

Downsides:

- can be unstable even if ODE is stable
- can fail

Demo sparse matrices

Finite difference method

Idea: Replace u' and u'' with finite differences.

For example: second-order centered

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} + O(h^2)$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

Demo

What happens for a nonlinear ODE?

Collocation method

LEFT OUT

ODE: $(*)$ $y'(x) = f(y(x))$

BC: $(**)$ $g(y(a), y(b)) = 0$

Step 1: Pick a basis (example: Chebyshev polynomials)

unknowns

$$v(x) = \sum_{i=1}^{n} \alpha_i T_i(x)$$

Goal: $v(x) \approx u(x)$

[Can't enforce this. Why?

Step II: Demand that $(*)$, $(**)$ satisfied at some points $x_1, \dots, x_n$

$\rightarrow$ (non-) linear system

Example: $u''(x) = 3x \quad u(0) = 2 \quad u(1) = 4$ (Still Left out)

Step 1: Pick a basis

$$v(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

Step II: Enforce ODE

$$v'(x) = \alpha_1 + 2\alpha_2 x + 3\alpha_3 x^2$$
$$v''(x) = 2\alpha_2 + 6\alpha_3 x$$

0 [where?

Pick collocation points.

0 [How many?

$$v''\left(\tfrac{1}{3}\right) = 2\alpha_2 + 6\alpha_3 \tfrac{1}{3} = 3 \cdot \tfrac{1}{3}$$
$$v''\left(\tfrac{2}{3}\right) = 2\alpha_2 + 6\alpha_3 \tfrac{2}{3} = 3 \cdot \tfrac{2}{3}$$

Step III: Enforce BCs

$$v(0) = \alpha_0 = 2$$
$$v(1) = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 4$$

Galerkin / Finite Element method

Book: least squares, LEFT OUT

given RHS

Example: $u''(x) = f(x)$ $\qquad u(a) = u(b) = 0$

Step I: Pick basis.

unknowns

$$v(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x)$$

Step II: Find residual.

$$0 \overset{!}{\approx} r(x) := v''(x) - f(x) = \sum_{i=1} \alpha_i \varphi_i''(x) - f(x)$$

won't be exactly satisfied. So: satisfy in a restricted sense.

Step III: Demand that residual is orthogonal to some test functions.

$$\int_a^b r(x)\, \psi_j(x)\, dx = 0 \qquad j = 1 \ldots n$$

How many?
$\underbrace{\qquad}_{\bigcirc}$

What to pick? Recycle basis! $\psi_j := \varphi_j$ "Galerkin method"

Have: $\int_a^b r(x)\,\varphi_j(x)\,dx = \int_a^b v''(x)\,\varphi_j(x)\,dx - \int_a^b f(x)\,\varphi_j(x)\,dx \overset{!}{=} 0$

Two derivatives? Ugh. Idea: Integrate by parts!

$$\int_a^b v''(x)\,\varphi_j(x)\,dx = \underbrace{\left[v'(x)\,\varphi_j(x)\right]_a^b}_{=0 \text{ if } \varphi_j(a)=\varphi_j(b)=0} - \int v'(x)\,\varphi_j'(x)\,dx$$

Get:

$$-\int_a^b v'(x)\,\varphi_j'(x_j)\,dx = \int f(x)\,\varphi_j(x)\,dx$$

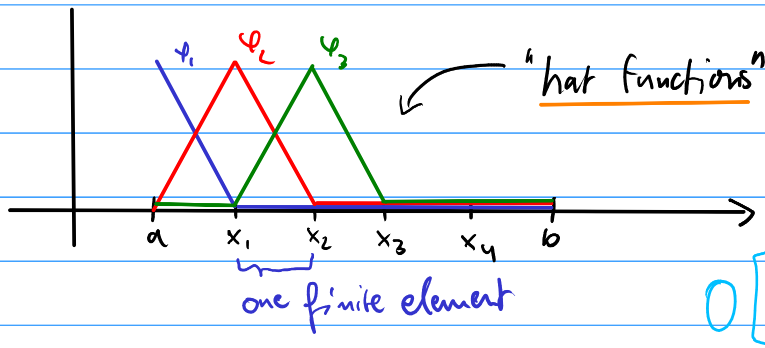
Plug in basis for v:

$$\int_a^b \left(\sum_i \alpha_i \varphi_i'(x)\right)\varphi_j'(x)\,dx = \int f(x)\,\varphi_j(x)$$

$$\sum_i \underbrace{\left(\int_a^b \varphi_i'(x)\,\varphi_j'(x)\,dx\right)}_{S_{ji}\ \text{"stiffness matrix"}} \alpha_i = \int f(x)\,\varphi_j(x)\,dx$$

Pick basis:

- could use "global" basis (eg. Chebyshev polynomials)
- or so-called "finite element" basis



Why these?

Now: Can compute stiffness mat, RHS, solve! \\"/