Numerical Methods for Partial Differential Equations

CS555 / MATH555 / CSE510

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Outline

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What's the point of this class?

PDEs describe lots of things in nature:

Idea: Use them to

Survey

- Home dept
- Degree pursued
- Longest program ever written
  - in Python?
- Research area

Class web page

https://bit.ly/numpde-s20

- Book Draft
- Notes, Class Outline
- Assignments (submission and return)
- Piazza
- Grading Policies/Syllabus
- Video
- Scribbles
- Demos (binder)

Sources for these Notes

- Adler, James, Hans De Sterck, Scott MacLachlan, and Luke N. Olson. *Numerical Partial Differential Equations*, 2020. (draft)
- Strikwerda, John C. *Finite Difference Schemes and Partial Differential Equations*, Second Edition. Other Titles in Applied Mathematics. Society for Industrial and Applied Mathematics, 2004.
- LeVeque, Randall J. *Numerical Methods for Conservation Laws*. 2nd ed. Birkhäuser Basel, 1992.
- Braess, Dietrich. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press, 2007.
- Shu, Chi-Wang. *Lecture Notes for AM257*, Brown University, Fall 2006.
- Heuveline, Vincent. *Lecture Notes for "Numerik für PDEs"*. Universität Karlsruhe, Summer 2005.
- Various prior bits of material by Luke Olson and Stephen Bond.

Open Source <3

These notes (and the accompanying demos) are open-source!

Bug reports and pull requests welcome:
https://github.com/inducer/numpde-notes

Outline

PDEs: Example I

What does this do? $\partial_t u = \partial_x u$

PDEs: Example II

What does this do? $\partial_x^2 u + \partial_y^2 u = 0$

Some good questions

- What is a time-like variable? (Variables labeled t?)
- What if there are boundaries?
  - In space?
  - In time?
- Existence and Uniqueness of Solutions?
  - Depends on where we look (the *function space*)
  - In the case of the two examples? (if there are no boundaries?)

Some general takeaways:

PDEs: An Unhelpfully Broad Problem Statement

Looking for $u : \Omega \to R^n$ where $\Omega \subseteq \mathbb{R}^d$ so that $u \in V$ and

$$F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots, x, y, \dots) = 0$$

Notation

Used as convenient:

$$u_x = \partial_x u = \frac{\partial u}{\partial x}$$

Properties of PDEs

What is the order of the PDE?

When is the PDE linear?

When is the PDE quasilinear?

When is the PDE semilinear?

Examples: Order, Linearity?

$$(xu^2)u_{xx} + (u_x + y)u_{yy} + u_x^3 + yu_y = f$$

$$(x + y + z)u_x + (z^2)u_y + (\sin x)u_z = f$$

Properties of Domains

May influence existence/uniqueness of solutions!

Function Spaces: Examples

Name some function spaces with their norms.

May *also* influence existence/uniqueness of solutions!

Solving PDEs

Closed-form solutions:

- If separation of variables applies to the domain: good luck with your ODE
- If not: Good luck! $\rightarrow$ Numerics

General Idea (that we will follow some of the time)

- Pick $V_h \subseteq V$ finite-dimensional
  - h is often a *mesh spacing*
- Approximate u through $u_h \in V_h$
- Show: $u_h \to u$ (in some sense) as $h \to 0$

Example

About grand big unifying theories

Is there a grand big unifying theory of PDEs?

Collect some stamps

$$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=g(x,y)$$

Discriminant value	Kind	Example
$b^2 - ac < 0$	Elliptic	Laplace $u_{xx} + u_{yy} = 0$
$b^2 - ac = 0$	Parabolic	Heat $u_t = u_{xx}$
$b^2 - ac > 0$	Hyperbolic	Wave $u_{tt} = u_{xx}$

Where do these names come from?

PDE Classification in Other Cases

Scalar first order PDEs?

First order systems of PDEs?

Classification in higher dimensions

$$Lu := \sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \text{lower order terms}$$

Consider the matrix $A(x) = (a_{ij}(x))_{i,j}$. May assume A symmetric. Why?

What cases can arise for the eigenvalues?

Elliptic PDE: Laplace/Poisson Equation

$$\triangle u = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u(x) \overset{\text{2D}}{=} u_{xx} + u_{yy} = f(x) \quad (x \in \Omega)$$

Called Laplace equation if $f = 0$. With Dirichlet boundary condition

$$u(x) = g(x) \qquad (x \in \partial\Omega).$$

Demo: Elliptic PDE Illustrating the Maximum Principle

Elliptic PDEs: Singular Solution

Demo: Elliptic PDE Radially Symmetric Singular Solution

Given $G(x) = C\log(|x|)$ as the free-space Green's function, can we construct the solution to the PDE with a more general f?

What can we learn from this?

Elliptic PDEs: Justifying the Singular Solution

$$u(x) = (G * f)(x) = \int_{\mathbb{R}^d} G(x-y)f(y)dy$$

Why?

Parabolic PDE: Heat Equation · Separation of Variables

$$\begin{aligned} u_t &= u_{xx} && ((x,t) \in [0,1] \times [0,T]) \\ u(x,0) &= g(x) && (x \in [0,1]) \\ u(0,t) = u(1,t) &= 0 && (t \in [0,T]) \end{aligned}$$

Parabolic PDE: Solution Behavior

Demo: Parabolic PDE What can we learn from analytic and numerical solution?

Hyperbolic PDE: Wave Equation

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx} \qquad && ((x,t) \in \mathbb{R} \times [0, T]) \\
u(x,0) &= g(x) \qquad && (x \in \mathbb{R})
\end{aligned}
$$

with $g(x) = \sin(\pi x)$.

Is this problem well-posed?

Can be rewritten in conservation law form:

Hyperbolic Conservation Laws

$$q_t(x, t) + \nabla \cdot \boldsymbol{F}(q(x, t)) = s(x)$$

Why is this called a conservation law?

$F :? \to ?$

Wave Equation as a Conservation Law

Rewrite the wave equation in conservation law form:

Solving Conservation Laws

Solve

$$u_t = v_x$$
$$v_t = u_x.$$

Demo: Hyperbolic PDE

Hyperbolic: Solution Properties

Properties of the solution for hyperbolic equations:

Outline

Interpolation and Vandermonde Matrices

Finite Differences Numerically

Demo: Finite Differences
Demo: Finite Differences vs Noise
Demo: Floating point vs Finite Differences

Taking Derivatives Numerically

Why *shouldn't* you take derivatives numerically?

Demo: Taking Derivatives with Vandermonde Matrices

Differencing Order of Accuracy Using Taylor

Find the order of accuracy of the finite difference formula
$f'(x) \approx [f(x+h) - f(x-h)]/2h$.

Outline

Interpolation Error

If f is n times continuously differentiable on a closed interval I and $p_{n-1}(x)$ is a polynomial of degree at most n that interpolates f at n distinct points $\{x_i\}$ $(i = 1, ..., n)$ in that interval, then for each x in the interval there exists ξ in that interval such that

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)(x - x_2)\cdots(x - x_n).$$

Set the error term to be $R(x) := f(x) - p_{n-1}(x)$ and set up an auxiliary function:

$$Y(t) = R(t) - \frac{R(x)}{W(x)}W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n}(t - x_i).$$

Note also the introduction of t as an additional variable, independent of the point x where we hope to prove the identity.

Interpolation Error: Proof cont'd

$$Y(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n} (t - x_i)$$

- Since x_i are roots of $R(t)$ and $W(t)$, we have $Y(x) = Y(x_i) = 0$, which means Y has at least $n+1$ roots.
- From Rolle's theorem, $Y'(t)$ has at least n roots, then $Y^{(n)}$ has at least one root ξ, where $\xi \in I$.
- Since $p_{n-1}(x)$ is a polynomial of degree at most $n-1$, $R^{(n)}(t) = f^{(n)}(t)$. Thus

$$Y^{(n)}(t) = f^{(n)}(t) - \frac{R(x)}{W(x)} n!.$$

- Plugging $Y^{(n)}(\xi) = 0$ into the above yields the result.

Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

- The error bound suggests choosing the interpolation nodes such that the product $|\prod_{i=1}^{n}(x - x_i)|$, is as small as possible. The Chebyshev nodes achieve this.
- Error is zero at the nodes
- If nodes scoot closer together near the interval ends, then

$$(x - x_1)(x - x_2)\cdots(x - x_n)$$

clamps down the (otherwise quickly-growing) error there.

Error Result: Simplified From

Boil the error result down to a simpler form.

Assume $x_1 < \cdots < x_n$.

- $\left|f^{(n)}(x)\right| \le M$ for $x \in [x_1, x_n]$,
- Set the interval length $h = x_n - x_1$.
  Then $|x - x_i| \le h$.

Altogether–there is a constant C independent of h so that:

$$\max_x |f(x) - p_{n-1}(x)| \le CMh^n.$$

For the grid spacing $h \to 0$, we have

$$E(h) = O(h^n).$$

This is called *convergence of order n*.

Demo: Interpolation Error

Outline

Outline

1D Advection Equation and Characteristics

$$u_t + au_x = 0, \quad u(0,x) = g(x) \qquad (x \in \mathbb{R})$$

Solution?

Solving Advection with Characteristics

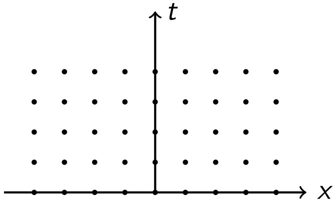
$$u_t + a u_x = 0, \quad u(0, x) = g(x) \qquad (x \in \mathbb{R})$$

Find the characteristic curve for advection.

Generalize this to a solution formula.

Does the solution formula admit solutions that aren't obviously allowed by the PDE?

Finite Difference for Hyperbolic: Idea



$$\{(x_k, t_\ell) : x_k = kh_x, t_\ell = \ell h_t\}$$

If $u(x, t)$ is the exact solution, want

$$u_{k,\ell} \approx u(x_k, t_\ell).$$

Condition at each grid point?

What are explicit/implicit schemes?

Designing Stencils

ETCS:

ITCS:

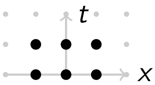
ETFS:

ETBS:

Terminology?

Write out ITCS:

Crank-Nicolson



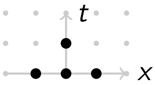
Crank-Nicolson

Write out Crank-Nicolson:

Lax-Wendroff

What's the core idea behind Lax-Wendroff?

Write out Lax-Wendroff.



Lax-Wendroff

Exploring Advection Schemes

Demo: Methods for 1D Advection

- Which of the schemes "work"?
- Any restrictions worth noting?

Outline

A Matrix View of Two-Level Stencil Schemes

Define

$$\mathbf{v}_\ell = \begin{bmatrix} u_{1,\ell} \\ \vdots \\ u_{N_x,\ell} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{N_t} \end{bmatrix}.$$

Define

$$\mathbf{u}_\ell = \begin{bmatrix} u(x_1, t_\ell) \\ \vdots \\ u(x_{N_x}, t_\ell) \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{N_t} \end{bmatrix}.$$

Definition (Two-Level Finite Difference Scheme)

A finite difference scheme that can be written as

is called a two-level linear finite difference scheme.

Rewriting Schemes in Matrix Form (1/2)

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell + h_t \boldsymbol{b}_\ell$$

Find P_h and Q_h for ETCS:

Rewriting Schemes in Matrix Form (2/2)

Find P_h and Q_h for Crank-Nicolson:

Truncation Error

Definition (Truncation Error)

Demo: Truncation Error Analysis via sympy

Error and Error Propagation

Express truncation error in our two-level framework:

Define $\boldsymbol{e}_\ell = \boldsymbol{u}_\ell - \boldsymbol{v}_\ell$. Understand the error as accumulation of truncation error:

Discrete and Continuous Norms

To measure properties of numerical solutions we need norms. Define a discrete L^∞ norm.

Define a discrete L^2 norm.

Important features:

Consistency and Convergence

Assume $u, (\partial_x^{q_x})u, (\partial_t^{q_t})u \in L^2(\mathbb{R} \times [0, t^*])$.

Definition (Consistency)

A two-level scheme is consistent in the L^2-norm with order q_t in time and q_x in space if

Definition (Convergence)

A two-level scheme is convergent in the L^2-norm with order q_t in time and q_x in space if

Analyzing ETFS

$$\frac{u_{k,\ell+1} - u_{k,\ell}}{h_t} + a\frac{u_{k+1,\ell} - u_{k,\ell}}{h_x} = 0$$

Let's understand more precisely what happens for this scheme.

ETFS Part 2

$$u_{k,\ell+1} = (1+\lambda)u_{k,\ell} - \lambda u_{k+1,\ell}$$

Consider $u(x,0) = 1_{[-1,0]}(x)$. Predict solution behavior.

Demo: Methods for 1D Advection (Revisit ETFS)

Stability

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell$$

Write down a matrix product to bring $\boldsymbol{v}_0$ to $\boldsymbol{v}_\ell$:

Definition (Stability)

A two-level scheme is stable in the L^2-norm if there exists a constant $c > 0$ independent of h_t and h_x so that

$$\left\| (P_h^{-1} Q_h)^\ell P_h^{-1} \right\| \leq c$$

for all ℓ and h_t such that $\ell h_t \leq t^*$.

Lax Convergence Theorem

Theorem (Lax Convergence)

If a two-level FD scheme is

- *consistent in the L^2-norm with order q_t in time and q_x in space, and*
- *stable in the L^2-norm, then*

it is convergent in the L^2-norm with order q_t in time and q_x in space.

Lax Convergence: Proof (1/2)

Lax Convergence: Proof (2/2)

$$\boldsymbol{e}_\ell = h_t \sum_{m=1}^{\ell} (P_h^{-1} Q_h)^{\ell-m} P_h^{-1} \boldsymbol{\tau}_{m-1}.$$

Conditions for Stability

$$\left\|(P_h^{-1}Q_h)^{\ell}P_h^{-1}\right\| \leq c$$

Give a simpler, sufficient condition:

How can we show bounds on these matrix norms?

Stability of ETBS (1/3)

Theorem (Gershgorin)

For a matrix $A \in \mathbb{C}^{N\times N} = (a_{i,j})$,

$$\sigma(A) \subset \bigcup_{j=1}^{N} \bar{B}\left(a_{j,j}, \sum_{k\neq j} |a_{j,k}|\right).$$

ETBS:

$$\frac{u_{k,\ell+1} - u_{k,l}}{h_t} + a\frac{u_{k,\ell} - u_{k-1,\ell}}{h_x} = 0$$

Analyze stability of ETBS:

Stability of ETBS (2/3)

$P_h = I$ and $Q_h = \text{tridiag}(\lambda, 1 - \lambda, 0)$.

Stability of ETBS (3/3)

Summarize ETBS stability:

Comments?

Outline

Discrete Time Fourier Transform

Assume $\boldsymbol{x}$ infinitely long. Define:

$$\hat{\boldsymbol{x}}(\theta) = \sum_k x_k e^{-i\theta k}$$

When is this well-defined?

Inverting the Fourier Transform

To recover $\boldsymbol{x}$:

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\boldsymbol{x}}(\theta) e^{i\theta k} d\theta.$$

Proof?

Getting to L^2

- Fourier Transform well defined for $\boldsymbol{x} \in \ell^1$.
- Problem: We care about L^2, not ℓ^1.

Theorem (Parseval)

If $\|\boldsymbol{x}\|_2 < \infty$, *then*

$$\|\boldsymbol{x}\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\boldsymbol{x}}(\theta)|^2 \, d\theta < \infty.$$

Impact?

Toeplitz Operators

Definition (Toeplitz Operator)

An operator T is a Toeplitz operator if $(T\boldsymbol{x})_j = \sum_k x_k p_{j-k}$. In this case, $\boldsymbol{p}$ is called the Toeplitz vector.

Example: ETCS

Let $\lambda = ah_t/2h_x$. Then

$$u_{k,\ell+1} = \lambda u_{k-1,\ell} + u_{k,l} - \lambda u_{k+1,\ell}.$$

Is ETCS Toeplitz?

Is ETCS Toeplitz?

$$(P_h \boldsymbol{u}_{\ell+1})_j = u_{j,\ell+1} \stackrel{!}{=} \sum_k u_{k,\ell+1} p_{j-k}$$

$$(Q_h \boldsymbol{u}_{\ell})_j = \lambda u_{k-1,\ell} + u_{k,l} - \lambda u_{k+1,\ell} \stackrel{!}{=} \sum_k u_{k,\ell} q_{j-k}$$

Fourier Transforms of Toeplitz Operators (1/3)

$$y_j = \sum_k x_k p_{j-k}$$

Fourier Transforms of Toeplitz Operators (2/3)

$$\hat{\boldsymbol{y}}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{\boldsymbol{x}}(\theta) \sum_j \left(\sum_k e^{i\varphi(k-j)} p_{j-k} \right) e^{i(\varphi-\theta)j} d\theta.$$

Fourier Transforms of Toeplitz Operators (3/3)

$$\hat{\boldsymbol{y}}(\theta) = \int_{-\pi}^{\pi} \hat{\boldsymbol{x}}(\theta)\hat{\boldsymbol{p}}(\varphi)\frac{1}{2\pi}\sum_{j} e^{i(\varphi-\theta)j} d\theta.$$

Fourier Transforms of Inverse Toeplitz Operators

Fourier transform $P_h^{-1} Q_h \boldsymbol{y}$?

Bounding the Operator Norm

Bound $\left\| P_h^{-1} Q_h \right\|_2^2$ using Fourier:

Is the upper bound attained?

von Neumann Stability

Two-level finite difference scheme

$$P_h \boldsymbol{v}_{\ell+1} = Q_h \boldsymbol{v}_\ell + h_t \boldsymbol{b}_\ell,$$

where P_h and Q_h are Toeplitz operators with vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

Definition (Symbol of a Two-Level Finite Difference Scheme)

Let

$$\hat{\boldsymbol{p}}(\theta) = \sum_k p_k e^{-i\varphi k}, \qquad \hat{\boldsymbol{q}}(\theta) = \sum_k q_k e^{-i\varphi k}.$$

Then the symbol of the two-level FD method is $s(\varphi) = \hat{\boldsymbol{q}}(\varphi)/\hat{\boldsymbol{p}}(\theta)$.

Definition (Von Neumann Stability)

If

$$\max_\varphi |s(\varphi)| \leq 1, \qquad \max_\varphi \left| \frac{1}{\hat{\boldsymbol{p}}(\varphi)} \right| \leq c$$

for some constant $c > 0$, we say the scheme is von Neumann stable.

Comparison with Lax-Richtmyer Stability

Need $\left\|(P_h^{-1}Q_h)^\ell P_h^{-1}\right\| \leq c$.

Why is bounding the symbol the most salient part?

Main restriction of von Neumann stability?

von Neumann Stability: ETBS (1/2)

ETBS: Let $\lambda = ah_t/h_x$. $u_{k,\ell+1} = \lambda u_{k-1,\ell} + (1-\lambda)u_{k,\ell}$.

von Neumann Stability: ETBS (2/2)

Found: $|s(\varphi)|^2 = 1 + 2(\lambda - \lambda^2)(\cos\varphi - 1)$.

von Neumann Stability: ETCS

Let $\lambda = ah_t/h_x$. Then

$$u_{k,\ell+1} = \frac{\lambda}{2} u_{k-1,\ell} + u_{k,\ell} - \frac{\lambda}{2} u_{k+1,\ell}.$$

von Neumann Stability: Crank-Nicolson

Let $\lambda = ah_t/(4h_x)$

$$-\lambda u_{k-1,\ell+1} + u_{k,\ell+1} + \lambda u_{k+1,\ell+1} = \lambda u_{k-1,\ell} + u_{k,\ell} - \lambda u_{k+1,\ell}.$$

Outline

Studying Solutions of the PDE

Saw numerically: interesting dispersion/dissipation behavior.
Want: theoretical understanding.

Consider *linear*, *continuous* (not yet discrete) differential operators

$$
\begin{aligned}
L_1 u &= u_t + a u_x, \\
L_2 u &= u_t - D u_{xx} + a u_x \qquad \text{(D>0)} \\
L_3 u &= u_t + a u_x - \mu u_{xxx}.
\end{aligned}
$$

What could we use as a prototype solution?

A Prototype Solution of the PDE

Observation: all these operators are diagonalized by complex exponentials. Come up with a 'prototype complex exponetial solution'.

What type of function is this?

Wave-like Solutions of the PDE

$$z(x,t) = z_0 e^{i(kx - \omega t)}$$

Observations in connection with L?

What is the dispersion relation?

Picking Apart the Dispersion Relation

Consider $\omega(k) = \alpha(k) + i\beta(k)$. Rewrite the wave solution with this.

How can we recognize dissipation?

What is the phase speed? How can we recognize dispersion?

Dispersion Relation: Examples

In each case, find the dispersion relation and identify properties.

$L_1 u = u_t + a u_x$

$L_2 u = u_t - D u_{xx} + a u_x \ (D > 0)$

$L_3 u = u_t + a u_x - \mu u_{xxx}$

Numerical Dissipation/Dispersion Analysis

Goal: Want discrete finite difference scheme to match dissipation/dispersion behavior of continuous PDE.

Define a discrete wave-like function:

We want $\boldsymbol{z}$ to solve $P_h \boldsymbol{z}_{\ell+1} = Q_h \boldsymbol{z}_{\ell}$. How can we connect the operators to the wave solution?

Toeplitz and Waves

$$z_{j,\ell} = z_0 e^{i(kjh_x - \omega \ell h_t)}.$$

Theorem (Waves Diagonalize Toeplitz Operators)

Let T be a Toeplitz operator. Then $T\boldsymbol{z}_\ell = \lambda(k)\boldsymbol{z}_\ell = \hat{\boldsymbol{t}}(kh_x)\boldsymbol{z}_\ell$.

Waves and Two-Level Schemes

Since P_h and Q_h are Toeplitz, we must have

$$P_h \boldsymbol{z}_{\ell+1} = \lambda_P(k)\boldsymbol{z}_{\ell+1}, \qquad Q_h \boldsymbol{z}_\ell = \lambda_Q(k)\boldsymbol{z}_\ell.$$

What does that mean?

Seen before?

Discrete Dispersion Relation (1/2)

So $\boldsymbol{z}_\ell$ is a solution of the finite difference scheme if $\omega = \omega(kh_x)$ satisfies

$$e^{-i\omega(\kappa)h_t} = s(\kappa),$$

where we let $\kappa = kh_x$. Interpret κ.

Let $s(\kappa) = |s(\kappa)|\, e^{i\varphi(\kappa)} = e^{\log|s(\kappa)| + i\varphi(\kappa)}$. $\omega(\kappa)$?

Discrete Dispersion Relation (2/2)

$$\omega(\kappa) = \frac{-\varphi(\kappa) + i \log |s(\kappa)|}{h_t}.$$

Plug that into the wave-like solution:

Criterion for stability?

Numerical Dispersion/Dissipation

Finite difference scheme $P_h \boldsymbol{u}_{\ell+1} = Q_h \boldsymbol{u}_\ell$ with symbol $s(k)$.

$$z_{j,\ell} = z_0 e^{\log|s(\kappa)|\ell} e^{ik\left(jh_x - \frac{-\varphi(\kappa)}{kh_t}\ell h_t\right)}$$

When is the scheme dissipative?

What is the phase speed?

Dispersion?

Dispersion/Dissipation Analysis of ETBS

Let $\lambda = ah_t/h_x$. Shown earlier: $s(kh_x) = 1 - \lambda(1 - e^{-ikh_x})$.

Dispersion/Dissipation Analysis of ETBS: Fine Grid

$$e^{-i\omega(\kappa)h_t} = 1 - \lambda(1 - e^{-ikh_x})$$

Dispersion/Dissipation: Demo

- **Demo:** Experimenting with Dispersion and Dissipation
- **Demo:** Dispersion and Dissipation

Outline

Heat Equation

Heat equation ($D > 0$):

$$\begin{aligned} u_t &= Du_{xx}, \qquad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= g(x) \qquad x \in \mathbb{R}. \end{aligned}$$

Fundamental solution ($g(x) = \delta(x)$):

Why is this a weird model?

Schemes for the Heat Equation

Cook up some schemes for the heat equation.

Explicit Euler:

Implicit Euler:

Von Neumann Analysis of Explicit Euler for Heat (1/2)

Let $\lambda = Dh_t/h_x^2$.

$$u_{k,\ell+1} = u_{k,\ell} + \lambda(u_{k+1,\ell} - 2u_{k,\ell} + u_{k-1,\ell}).$$

Von Neumann Analysis of Explicit Euler for Heat (2/2)

$$-2 \le 2\lambda(\cos(\varphi) - 1) \le 0.$$

Comment on the stability region found regarding speeds of propagation.

Von Neumann Analysis of Implicit Euler for Heat

Let $\lambda = Dh_t/h_x^2$.

$$u_{k,\ell+1} - \lambda(u_{k+1,\ell+1} - 2u_{k,\ell+1} + u_{k-1,\ell+1}) = u_{k,\ell}$$

Does the type of system we need to solve for implicit+parabolic correspond to another PDE?

Outline

Outline

Conservation Laws: Recap

$$u_t + f(u)_x = 0,$$

where u is a function of x and $t \in \mathbb{R}_0^+$.

Rewrite in integral form:

Recall: Characteristic Curve: a function $x(t)$ so that $u(x(t), t) = u(x_0, 0)$.

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = f'(u(x(t), t)), \\ x(0) = x_0. \end{cases}$$

What assumption underlies all this?

Burger's Equation

Consider Burgers' Equation:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \\ u(x,0) = g(x) = \sin(x). \end{cases}$$

Interpret Burger's equation.

Consider the characteristics at $\pi/2$ and $3\pi/2$.

Weak Solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t)\mathrm{d}x = f(u(a,t)) - f(u(b,t))$$

Define a weak solution:

Rankine-Hugoniot Condition (1/2)

Consider: Two C^1 segments separated by a curve $x(t)$ with no regularity.

Rankine-Hugoniot Condition (2/2)

$$(d/dt)G_a(x(t),t) = u(x(t),t)x'(t) - (f(u(x(t),t)) - f(u(a,t))).$$

Rankine-Hugoniot and Weak Solutions

Theorem (Rankine-Hugoniot and Weak Solutions)

If u is piecewise C^1 and is discontinuous only along isoated curves, and if u satisfies the PDE when it is C^1, and the Rankine-Hugoniot condition holds along all discontinuous curves, then u is a weak solution of the conservation law.

Riemann Problems: Example 1

Consider the following Riemann problem:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = \begin{cases} 1 & x < 0, \\ -1 & x \geq 0. \end{cases}$$

Riemann Problems: Example 2

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = \begin{cases} -1 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

(IC sign flip compared to previous slide)

Bad Shocks and Good Shocks

In the shock version of the ‘ambiguous’ Riemann problem, where do the characteristics go?

Comment on the stability of that situation.

Ad-Hoc Idea: Ban Bad Shocks

Recall: what is $f'(u)$?

Devise a way to ban unstable shocks.

Vanishing Viscosity Solutions

Goal: neither uniqueness nor existence poses a problem.

How?

Entropy-Flux Pairs

What are features of (physical) entropy?

Definition (Entropy/Entropy Flux)

An entropy $\eta(u)$ and an entropy flux $\psi(u)$ are functions so that η is convex and

$$\eta(u)_t + \psi(u)_x = 0$$

for smooth solutions of the conservation law.

Finding Entropy-Flux Pairs

$\eta(u)_t + \psi(u)_x = 0$. Find conditions on η and ψ.

Come up with an entropy-flux pair for Burgers.

Back to Vanishing Viscosity (1/2)

$$u_t + f(u)_x = \varepsilon u_{xx}$$

What's the evolution equation for the entropy?

Back to Vanishing Viscosity (2/2)

$$\eta(u)_t + \psi(u)_x = \varepsilon(\eta'(u)u_x)_x - \varepsilon\eta''(u)u_x^2.$$

Integrate this over $[x_1, x_2] \times [t_1, t_2]$.

Entropy Solution

Definition (Entropy solution)

The function $u(x, t)$ is the entropy solution of the conservation law if for all convex entropy functions and corresponding entropy fluxes, the inequality

$$\eta(u)_t + \psi(u)_x \leq 0$$

is satisfied in the weak sense.

Conservation of Entropy?

What can you say about conservation of entropy in time?

Total Variation

$$\mathrm{TV}(u) = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int |u(x+\varepsilon) - u(x)| \, dx.$$

Simpler form if u is differentiable?

Hiking analog?

Total Variation and Conservation Laws

Theorem (Total Variation is Bounded [Dafermos 2016, Thm. 6.2.6])

Let u be a solution to a conservation law with $f''(u) \geq 0$. Then:

$$\mathrm{TV}(u(t+\Delta t,\cdot)) \leq \mathrm{TV}(u(t,\cdot)) \quad \textit{for } \Delta t \geq 0.$$

Theorem (L^1 contraction [Dafermos 2016, Thm. 6.3.2])

Let u, v be viscosity solutions of the conservation law. Then

$$\|u(t+\Delta t,\cdot) - v(t+\Delta t,\cdot)\| \leq \|u(t,\cdot) - v(t,\cdot)\| \quad \textit{for } \Delta t \geq 0$$

Outline

Finite Difference for Conservation Laws? (1/2)

$$\begin{cases} u_t + \left(\frac{u}{2}\right)^2_x = 0 \\ u(x,0) = \begin{cases} 1 & x < 0, \\ 0 & x \geq 0. \end{cases} \end{cases}$$

Entropy Solution?

Rewrite the PDE to ‘match’ the form of advection $u_t + au_x = 0$:

Equivalent?

Finite Difference for Conservation Laws? (2/2)

Recall the *upwind scheme* for $u_t + au_x = 0$:

Write the upwind FD scheme for $u_t + uu_x = 0$:

Schemes in Conservation Form

Definition (Conservative Scheme)

A conservation law scheme is called conservative iff it can be written as

where $f^* \ldots$

Theorem (Lax-Wendroff)

If the solution $\{u_{j,\ell}\}$ to a conservative scheme converges (as $\Delta t, \Delta x \to 0$)

Lax-Wendroff Theorem: Proof

Summation by parts: With $\Delta^+ a_k = a_{k+1} - a_k$ and $\Delta^- a_k = a_k - a_{k-1}$:

$$\sum_{k=1}^{N} a_k(\Delta^- \varphi_k) + \sum_{k=1}^{N} \varphi_k(\Delta^+ a_k) = -a_1\varphi_0 + \varphi_N a_{N+1}.$$

Finite Volume Schemes

Finite volume: Idea?

Developing Finite Volume

$$\int_{t_\ell}^{t^{\ell+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} (u_t + f(u)_x) \mathrm{d}x \mathrm{d}t = 0$$

Flux Integrals?

$$\frac{1}{h_x}\int_{t_\ell}^{t_{\ell+1}} f(u_{j+1/2})\mathrm{d}t?$$

The Godunov Scheme

Altogether:

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})).$$

Overall algorithm?

Heuristic time step restriction?

Riemann Problem

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = \begin{cases} u_l & x < 0, \\ u_r & x \geq 0 \end{cases} \end{cases}$$

Exact solution in the Burgers case?

Rieamnn Solver for a General Conservation Law

To complete the scheme: Need $f^*(u^-, u^+)$. For Burgers: already known. For a general (convex/concave-f) conservation law?

Equivalent to

$$f^*(u^-, u^+) = \begin{cases} \max_{u^+ \le u \le u^-} f(u) & \text{if } u^- > u^+, \end{cases}$$

More Riemann Solvers

Downside of Godunov Riemann solver?

Back to Advection

Consider only $f(u) = au$ for now. Riemann solver inspiration from FD?

Side Note: First Order Upwind, Rewritten

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f^*(u_{j,\ell}, u_{j+1,\ell}) - f^*(u_{j-1,\ell}, u_{j,\ell})}{h_x}$$

with

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-).$$

Lax-Friedrichs

Generalize linear upwind flux for a nonlinear conservation law:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{|a|}{2}(u^+ - u^-).$$

Demo: Finite Volume Burgers (Part I)

Outline

Improving Accuracy

Consider our existing discrete FV formulation:

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} - \frac{h_t}{h_x}(f(u_{j+1/2,\ell}) - f(u_{j-1/2,\ell})).$$

What obstacles exist to increasing the order of accuracy?

What order of accuracy can we expect?

Improving the Order of Accuracy

Improve temporal accuracy.

What's the obstacle to higher spatial accuracy?

How can we improve the accuracy of that approximation?

Increasing Spatial Accuracy

Temporary Assumptions:

- $f'(u) \geq 0$
- $f^*_{j+1/2} = f(\bar{u}_j)$ (e.g. Godunov in this situation)

Reconstruct $u_{j+1/2}$ using $\{\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}\}$. Accuracy? Names?

Compute fluxes, use increments over cell average:

Lax-Wendroff

For $u_t + au_x$, from finite difference:

$$f^*(u^-, u^+) = \frac{au^- + au^+}{2} - \frac{a^2}{2} \cdot \frac{\Delta t}{\Delta x}(u^+ - u^-).$$

Taylor in time: $u_{\ell+1} = u_\ell + \partial_t u_\ell \cdot h_t + \partial_t^2 u_\ell \cdot h_t/2 + O(h_t^3)$.

$$\frac{u_{j,\ell+1} - u_{j,\ell}}{h_t} + \frac{f(u_{j+1,\ell}) - f(u_{j-1,\ell})}{2h_x}$$
$$= \frac{h_t}{2h_x}\left[f'(u_{j+1/2,\ell})\frac{f(u_{j+1,\ell}) - f(u_{j,\ell})}{h_x} - f'(u_{j-1/2,\ell})\frac{f(u_{j,\ell}) - f(u_{j-1,\ell})}{h_x}\right]$$

As a Riemann solver:

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{\cdots} - \frac{h_t}{\cdots}[f'(u^\circ)(f(u^+) - f(u^-))].$$

Monotone Schemes

Definition (Monotone Scheme)

A scheme

$$\begin{aligned} u_{j,\ell+1} &= u_{j,\ell} - \lambda(f^*(u_{j-p},\ldots,u_{j+q}) - f^*(u_{j-p-1},\ldots,u_{j+q-1})) \\ &=: G(u_{j-p-1},\ldots,u_{j+q}) \end{aligned}$$

is called a montone scheme if G is a monotonically nondecreasing function $G(\uparrow,\uparrow,\ldots,\uparrow)$ of each argument.

Monotonicity for Three-Point Schemes

Three-Point Scheme:

$$G(u_{j-1}, u_j, u_{j+1}) = u_j - \lambda[f^*(u_j, u_{j+1}) - f^*(u_{j-1}, u_j)].$$

When is this monotone?

Lax-Friedrichs is Monotone

$$f^*(u^-, u^+) = \frac{f(u^-) + f(u^+)}{2} - \frac{\alpha}{2}(u^+ - u^-).$$

Show: This is monotone.

Monotone Schemes: Properties

Theorem (Good properties of monotone schemes)

- *Local maximum principle:*

$$\min_{i \in \textit{stencil around } j} u_i \leq G(u)_j \leq \max_{i \in \textit{stencil around } j} u_i .$$

- *L^1-contraction:*

$$\|G(u) - G(v)\|_{L^1} \leq \|u - v\|_{L^1} .$$

- *TVD:*

$$TV(G(u)) \leq TV(u).$$

- *Solutions to monotone schemes satisfy all entropy conditions.*

Godunov's Theorem

Theorem (Godunov)

Monotone schemes are at most first-order accurate.

What now?

Linear Schemes

Definition (Linear Schemes)

A scheme is called a linear scheme if it is linear when applied to a linear PDE:

$$u_t + au_x = 0,$$

where a is a constant.

Write the general case of a linear scheme for $u_t + u_x = 0$:

Linear + TVD = ?

Theorem (TVD for linear Schemes)

For linear schemes, TVD $\Rightarrow$ monotone.

What does that mean?

Now what?

Harten's Lemma

Theorem (Harten's Lemma)

If a scheme can be written as

$$\bar{u}_{j,\ell+1} = \bar{u}_{j,\ell} + \lambda(C_{j+1/2}\Delta_+\bar{u}_j - D_{j-1/2}\Delta_-\bar{u}_j)$$

with $C_{j+1/2} \geq 0$, $D_{j+1/2} \geq 0$, $1 - \lambda(C_{j+1/2} + D_{j+1/2}) \geq 0$ *and* $\lambda = h_t/h_x$, *then it is TVD.*

As a matter of notation, we have

$$\begin{aligned} \Delta_+ u_j &= u_{j+1} - u_j, \\ \Delta_- u_j &= u_j - u_{j-1}. \end{aligned}$$

We have omitted the time subscript for the time level ℓ.

Harten's Lemma: Proof

Minmod Scheme

Still assume $f'(u) \geq 0$.

$$f^{*,(1)}_{j+1/2} = f\left(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_{j+1} - \bar{u}_j)}_{\tilde{u}_j^{(1)}}\right), \qquad f^{*,(2)}_{j+1/2} = f\left(\bar{u}_j + \underbrace{\frac{1}{2}(\bar{u}_j - \bar{u}_{j-1})}_{\tilde{u}_j^{(2)}}\right).$$

Design a 'safe' thing to use for $\tilde{u}$:

Minmod is TVD

Show that Minmod is TVD:

Minmod: CFL restriction?

Derive a time step restriction for Minmod.

What about Time Integration?

$$u^{(1)} = u_\ell + h_t L(u_\ell), \qquad u_{\ell+1} = \frac{u_\ell}{2} + \frac{1}{2}(u^{(1)} + h_t L(u^{(1)})).$$

Above: A version of RK2 with L the ODE RHS. Will this cause wrinkles?

Total Variation is Convex

Show: $\mathrm{TV}(\cdot)$ is a convex functional.

TVD and High Order

Can TVD schemes be high order everywhere? (aside from near shocks)

High Order at Smooth Extrema

- TVB Schemes [Shu '87]
- ENO [Harten/Engquist/Osher/Chakravarthy '87]
  - Define $W_j = w(x_{j+1/2}) = \int_{x_{1/2}}^{x_{j+1/2}} u(\xi, t) d\xi = h_x \sum_{i=1}^{j} \bar{u}_i$
    - Observe $u_{j+1/2} = w'(x_{j+1/2})$.
    - Approximate by interpolation/numerical differentiation.
  - Start with the linear function $p^{(1)}$ through W_{j-1} and W_j
  - Compute divided differences on (W_{j-2}, W_{j-1}, W_j)
  - Compute divided differences on (W_{j-1}, W_j, W_{j+1})
  - Use the one with the smaller magnitude (of the divided differences) to extend $p^{(1)}$ to quadratic
  - (and so on, adding points on the side with the lowest magnitude of the divided differences)
- WENO [Liu/Osher/Chan '94]

Outline

Systems of Conservation Laws

Linear system of hyperbolic conservation laws, $A \in \mathbb{R}^{m \times m}$:

$$\begin{aligned} \boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\ \boldsymbol{u}(x,0) &= \boldsymbol{u}_0(x). \end{aligned}$$

Assumptions on A?

Linear System Solution

$$\boldsymbol{v} = R^{-1}\boldsymbol{u}, \qquad \boldsymbol{v}_t + \Lambda \boldsymbol{v}_x = 0.$$

Write down the solution.

What is the impact on boundary conditions? E.g. $(\lambda_p) = (-c, 0, c)$ for a BC at $x = 0$ for $[0, 1]$?

Characteristics for Systems (1/2)

Consider system $\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0$. Write in quasilinear form:

When hyperbolic?

Characteristics for Systems (2/2)

What about characteristics/shock speeds?

Are values of $\boldsymbol{u}$ still constant along characteristics?

Shocks and Riemann Problems for Systems

$$\begin{aligned} \boldsymbol{u}_t + A\boldsymbol{u}_x &= 0, \\ \boldsymbol{u}(x,0) &= \begin{cases} \boldsymbol{u}_l & x < 0, \\ \boldsymbol{u}_r & x > 0. \end{cases} \end{aligned}$$

Solution? (Assume strict hyperbolicity with $\lambda_1 < \lambda_2 < \cdots < \lambda_m$.)

Shock Fans (1/2)

What does the solution look like?

Jump across the characteristic associated with λ_p?

Shock Fans (2/2)

Do those jumps satisfy Rankine-Hugoniot?

How can we find intermediate values of $\boldsymbol{u}$?

Two Dimensions

$u_t + f(u)_x + g(u)_y = 0$. Finite volume methods generalize in principle:

However:

Outline

Outline

Function Spaces

Consider

$$f_n(x) = \begin{cases} -1 & x \leq -\frac{1}{n}, \\ \frac{3n}{2}x - \frac{n^3}{2}x^3 & -\frac{1}{n} < x < \frac{1}{n}, \\ 1 & x \geq 1/n. \end{cases}$$

Converges to the step function. Problem?

Norms

Definition (Norm)

A norm $\|\cdot\|$ maps an element of a *vector space* into $[0, \infty)$. It satisfies:

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Convergence

Definition (Convergent Sequence)

$x_n \to x :\Leftrightarrow \|x_n - x\| \to 0$ (convergence in norm)

Definition (Cauchy Sequence)

Banach Spaces

Definition (Complete/“Banach” space)

What’s special about Cauchy sequences?

Counterexamples?

More on C^0

Let $\Omega \subseteq \mathbb{R}^n$ be open. Is $C^0(\Omega)$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

Is $C^0(\bar{\Omega})$ with $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$ Banach?

C^m Spaces

Let $\Omega \subseteq \mathbb{R}^n$.

Consider a multi-index $\boldsymbol{k} = (k_1, \ldots, k_n)$ and define the symbols

Definition (C^m Spaces)

L^p Spaces

Let $1 \leq p < \infty$.

Definition (L^p Spaces)

$$L^p(\Omega) := \left\{ u : (u : \mathbb{R} \to \mathbb{R}) \text{ measurable}, \int_\Omega |u|^p \, dx < \infty \right\},$$

$$\|u\|_p := \left(\int_\Omega |u|^p \, dx \right)^{1/p}.$$

Definition (L^∞ Space)

$$L^\infty(\Omega) := \{ u : (u : \mathbb{R} \to \mathbb{R}), |u(x)| < \infty \text{ almost everywhere} \},$$

$$\|u\|_\infty = \inf \{ C : |u(x)| \leq C \text{ almost everywhere} \}.$$

L^p Spaces: Properties

Theorem (Hölder's Inequality)

For $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$ and measurable u and v,

Theorem (Minkowski's Inequality (Triangle inequality in L^p))

For $1 \leq p \leq \infty$ and $u, v \in L^p(\Omega)$,

Inner Product Spaces

Let V be a vector space.

Definition (Inner Product)

An inner product is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ such that for any $f, g, h \in V$ and $\alpha \in \mathbb{R}$

$$\begin{aligned}
\langle f, f\rangle &\geq 0, \\
\langle f, f\rangle &= 0 \Leftrightarrow f = 0, \\
\langle f, g\rangle &= \langle f, g\rangle, \\
\langle \alpha f + g, h\rangle &= \alpha \langle f, h\rangle + \langle g, h\rangle.
\end{aligned}$$

Definition (Induced Norm)

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

Hilbert Spaces

Definition (Hilbert Space)

An inner product space that is complete under the induced norm.

Let Ω be open.

Theorem (L^2)

$L^2(\Omega)$ equals the closure of (set of all imits of Cauchy sequences in) $C_0^\infty(\Omega)$ under the induced norm $\|\cdot\|_2$.

Theorem (Hilbert Projection)

Weak Derivatives

Define the space L^1_{loc} of locally integrable functions.

Definition (Weak Derivative)

$v \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of $u \in L^1_{\text{loc}}(\Omega)$ of multi-index order $\boldsymbol{k}$ if

Weak Derivatives: Examples (1/2)

Consider all these on the interval $[-1, 1]$.

$$f_1(x) = 4(1-x)x$$

$$f_2(x) = \begin{cases} 2x & x \leq 1/2, \\ 2-2x & x > 1/2. \end{cases}$$

Weak Derivatives: Examples (2/2)

$$f_3(x) = \sqrt{\frac{1}{2}} - \sqrt{|x - 1/2|}$$

Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}_0$ and $1 \leq p < \infty$.

Definition ((k, p)-Sobolev Norm/Space)

More Sobolev Spaces

$W^{0,2}$?

$W^{s,2}$?

$H_0^1(\Omega)$?

Outline

An Elliptic Model Problem

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^1(\Omega)$.

$$\begin{aligned} -\nabla \cdot \nabla u + u &= f(x) \quad (x \in \Omega), \\ u(x) &= 0 \qquad (x \in \partial\Omega). \end{aligned}$$

Let $V := H_0^1(\Omega)$. Integration by parts? (Gauss's theorem applied to $a\boldsymbol{b}$):

Weak form?

Motivation: Bilinear Forms and Functionals

$$\int_\Omega \nabla u \cdot \nabla v + \int_\Omega uv = \int fv.$$

This is the weak form of the strong-form problem. The task is to find a $u \in V$ that satisfies this for all test functions $v \in V$.

Recast this in terms of bilinear forms and functionals:

Dual Spaces and Functionals

Bounded Linear Functional

Let $(V, \|\cdot\|)$ be a Banach space. A linear functional is a linear function $g : V \to \mathbb{R}$. It is bounded ($\Leftrightarrow$ continuous) if there exists a constant C so that $|g(v)| \leq C \|v\|$ for all $v \in V$.

Dual Space

Let $(V, \|\cdot\|)$ be a Banach space. Then the dual space V' is the space of bounded linear functionals on V.

Dual Space is Banach (cf. e.g. Trèves 1967)

V' is a Banach space with the dual norm

Functionals in the Model Problem

Is g from the model problem a bounded functional? (In what space?)

That bound felt loose and wasteful. Can we do better?

Riesz Representation Theorem (1/3)

Let V be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Theorem (Riesz)

Let g be a bounded linear functional on V, i.e. $g \in V'$. Then there exists a unique $u \in V$ so that $g(v) = \langle u, v \rangle$ for all $v \in V$.

Riesz Representation Theorem: Proof (2/3)

Have $w \in N(g)^\perp \setminus \{0\}$, $\alpha = g(w) \neq 0$, and $z := v - (g(v)/\alpha)w \perp w$.

Riesz Representation Theorem: Proof (3/3)

Uniqueness of u?

Back to the Model Problem

$$
\begin{aligned}
a(u,v) &= \langle \nabla u, \nabla v\rangle_{L^2} + \langle u, v\rangle_{L^2} \\
g(v) &= \langle f, v\rangle_{L^2} \\
a(u,v) &= g(v)
\end{aligned}
$$

Have we learned anything about the solvability of this problem?

Poisson

Let $\Omega \subset \mathbb{R}^n$ open, bounded, $f \in H^{-1}(\Omega)$.

This is called the Poisson problem (with Dirichlet BCs).

Weak form?

Ellipticity

Let V be Hilbert space.

V-Ellipticity

A bilinear form $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ is called coercive if there exists a constant $c_0 > 0$ so that

and a is called continuous if there exists a constant $c_1 > 0$ so that

If a is both coercive and continuous on V, then a is said to be V-elliptic.

Lax-Milgram Theorem

Let V be Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Lax-Milgram, Symmetric Case

Let a be a V-elliptic bilinear form that is also symmetric, and let g be a bounded linear functional on V.
Then there exists a unique $u \in V$ so that $a(u, v) = g(v)$ for all $v \in V$.

Back to Poisson

Can we declare victory for Poisson?

Can this inequality hold in general, without further assumptions?

Poincaré-Friedrichs Inequality (1/3)

Theorem (Poincaré-Friedrichs Inequality)

Suppose $\Omega \subset \mathbb{R}^n$ is bounded and $u \in H_0^1(\Omega)$. Then there exists a constant $C > 0$ such that

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2} .$$

Poincaré-Friedrichs Inequality (2/3)

Prove the result in $C_0^\infty(\Omega)$.

Poincaré-Friedrichs Inequality (3/3)

Prove the result in $H_0^1(\Omega)$.

Back to Poisson, Again

Show that the Poisson bilinear form is coercive.

Draw a conclusion on Poisson:

Outline

Ritz-Galerkin

Some key goals for this section:

- How do we use the weak form to compute an approximate solution?
- What can we know about the accuracy of the approximate solution?

Can we pick one underlying principle for the construction of the approximation?

Galerkin Orthogonality

$$a(u, v) = g(v) \qquad \text{for all } v \in V, a(u_h, v_h) \quad = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Observations?

Céa's Lemma

Let $V \subset H$ be a closed subspace of a Hilbert space H.

Céa's Lemma

Let $a(\cdot,\cdot)$ be a coercive and continuous bilinear form on V. In addition, for a bounded linear functional g on V, let $u \in V$ satisfy

$$a(u,v) = g(v) \qquad \text{for all } v \in V.$$

Consider the finite-dimensional subspace $V_h \subset V$ and $u_h \in V_h$ that satisfies

$$a(u_h, v_h) = g(v_h) \qquad \text{for all } v_h \in V_h.$$

Then

Céa's Lemma: Proof

Recall Galerkin orthgonality: $a(u_h - u, v_h) = 0$ for all $v_h \in V_h$. Show the result.

Elliptic Regularity

Definition (H^s Regularity)

Let $m \geq 1$, $H_0^m(\Omega) \subseteq V \subseteq H^m(\Omega)$ and $a(\cdot,\cdot)$ a V-elliptic bilinear form. The bilinear form $a(u,v) = \langle f, v\rangle$ for all $v \in V$ is called H^s regular, if for every $f \in H^{s-2m}$ there exists a solution $u \in H^s(\Omega)$ and we have with a constant $C(\Omega, a, s)$,

Theorem (Elliptic Regularity (cf. Braess Thm. 7.2))

Let a be a H_0^1-elliptic bilinear form with sufficiently smooth coefficient functions.

Elliptic Regularity: Counterexamples

Are the conditions on the boundary essential for elliptic regularity?

Are there any particular concerns for mixed boundary conditions?

Estimating the Error in the Energy Norm

Come up with an idea of a bound on $\|u - u_h\|_{H^1}$.

What's still to do?

L^2 Estimates

Let H be a Hilbert space with the norm $\|\cdot\|_H$ and the inner product $\langle \cdot, \cdot \rangle$. (Think: $H = L^2$, $V = H^1$.)

Theorem (Aubin-Nitsche)

Let $V \subseteq H$ be a subspace that becomes a Hilbert space under the norm $\|\cdot\|_V$. Let the embedding $V \to H$ be continuous. Then we have for the finite element solution $u \in V_h \subset V$:

if with every $g \in H$ we associate the unique (weak) solution φ_g of the equation (also called the dual problem)

Aubin-Nitsche: Proof

L^2 Estimates using Aubin-Nitsche

$$\|u - u_h\|_H \leq c_1 \|u - u_h\|_V \sup_{g \in H} \left[\frac{1}{\|g\|_H} \inf_{v_h \in V_h} \|\varphi_g - v_h\|_V \right],$$

If $u \in H_0^1(\Omega)$, what do we get from Aubin-Nitsche?

So does Aubin-Nitsche give us an L^2 estimate?

Outline

Finite Elements in 1D: Discrete Form

$\Omega := [\alpha, \beta]$. Look for $u \in H_0^1(\Omega)$, so that $a(u, \varphi) = \langle f, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega)$. Choose $V_h = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$ and expand $u_h = \sum_{i=1}^n u_h^i \varphi_i \in V_h$. Find the discrete system.

Grids and Hats

Let $I_i := [\alpha_i, \beta_i]$, so that $\bar{\Omega} = \bigcup_{i=0}^{N} I_i$ and $I_i^\circ \cap I_j = \emptyset$ for $i \neq j$. Consider a grid

$$\alpha = x_0 < \cdots < x_N < x_{N+1} = \beta,$$

i.e. $\alpha_i = x_i$, $\beta_i = x_{i+1}$ for $i \in \{0, \ldots, N\}$. The $\{x_i\}$ are called nodes of the grid. $h_i := x_{i+1} - x_i$ for $i \in \{0, \ldots, N\}$ and $h := \max_i h_i$. V_h? Basis?

Degrees of Freedom and Matrices

Define something more general than basis coefficients to solve for.

Define shape functions and assemble the stiffness matrix:

A Matrix Property for Efficiency

$$(A_h)_{i,j} = a(\hat{\varphi}_j, \hat{\varphi}_i).$$

Anything special about the matrix?

Error Estimation

According to Céa, what's our main missing piece in error estimation now?

Interpolation Error (1D-only)

For $v \in H^2(\Omega)$,

If $v \in H^1(\Omega) \setminus H^2(\Omega)$,

Is I_h^1 defined for $v \in H^2$? for $v \in H^1 \setminus H^2$?

Interpolation Error: Towards an Estimate

Provide an a-priori estimate.

What's the relationship between $I_h^1 u$ and u_h?

Local-to-Global

Is there a simple way of constructing the polynomial basis?

Local-to-Global: Math

Construct a polynomial basis using this approach.

Demo

Demo: Developing FEM in 1D

Going Higher Order

$\Omega \subset \mathbb{R}$ with a grid as above.

Possible extension:

Higher Order Approximation

Let $0 \leq \ell \leq k$. Then for $v \in H^{\ell+1}(\Omega)$,

High-Order: Degrees of Freedom

Define some degrees of freedom (or DoFs) for high-order 1D FEM.

High-Order: Local Basis

Define local form functions for high-order 1D FEM.

High-Order: Global Basis

Obtain the global shape functions for high-order 1D FEM.

Outline

A Boundary Value Problem

Consider the following elliptic PDE

$$-\nabla \cdot (\kappa(\boldsymbol{x}) \nabla u) = f(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega \subset \mathbb{R}^2,$$
$$u(\boldsymbol{x}) = 0 \quad \text{when} \quad \boldsymbol{x} \in \partial\Omega.$$

Weak form?

Weak Form: Bilinear Form and RHS Functional

Hence the problem is to find $u \in V$, such that

$$a(u, v) = g(v), \quad \text{for all } v \in V = H_0^1(\Omega)$$

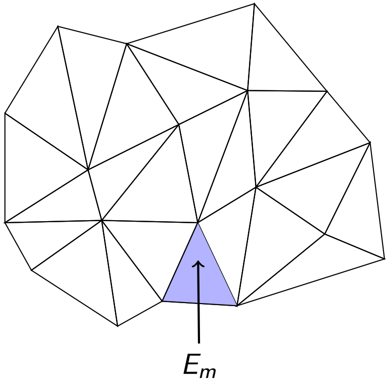
where...

Is this symmetric, coercive, and continuous?

Triangulation: 2D

Suppose the domain is a union of triangles E_m, with vertices x_i.

Ω



$$\bar{\Omega} = \bigcup_{i=1}^{M} E_m.$$

Elements and the Bilinear Form

If the domain, Ω, can be written as a disjoint union of elements, E_k,

$$\Omega = \cup_{m=1}^{M} E_m \quad \text{with} \quad E_i^\circ \cap E_j^\circ = \emptyset \text{ for } i \neq j,$$

what happens to a and g?

Basis Functions

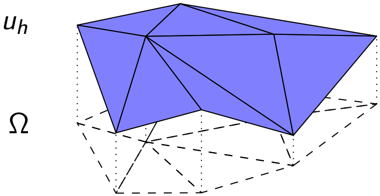
Expand

$$u_N(\boldsymbol{x}) = \sum_{i=1}^{N_p} u_i \varphi_i,$$

and plug into the weak form.

Global Lagrange Basis

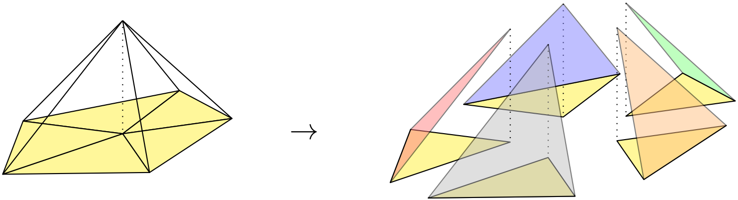
Approximate solution u_h: Piecewise linear on Ω



The Lagrange basis for V_h consists of piecewise linear φ_i, with...

Basis Functions Features

Features of the basis?



Local Basis

What basis functions exist on each triangle?

Local Basis Expressions

Write expressions for the nodal linear basis in 2D.

Higher-Order, Higher-Dimensional Simplex Bases

What's an n-simplex?

Give a higher-order polynomial space on the n-simplex:

Give nodal sets (on the $\triangle$) for P^N and $\dim P^N$ in general.

Finding a Nodal/Lagrange Basis in General

Given a nodal set $(\xi_i)_{i=1}^{N_p} \subset \hat{E}$ (where $\hat{E}$ is the reference element) and a basis $(\varphi_j)_{j=1}^{N_p} : \hat{E} \to \mathbb{R}$, find a Lagrange basis.

Higher-Order, Higher-Dimensional Tensor Product Bases

What's a tensor product element?

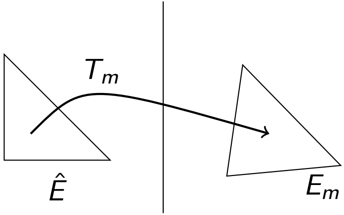
Give a higher-order polynomial space on the n-simplex:

Give the nodal sets (on the quad) for Q^N.

Tensor Product Elements: Lagrange Basis

Lagrange Basis for Tensor Product Elements?

Element Mappings



Construct a mapping $T_m : \hat{E} \to E_m$. Reference element $\hat{E}$, global $\triangle$ E_m.

What is the Jacobian of T_m?

More on Mappings

Is an affine mapping sufficient for a tensor product element?

How might we accomplish curvilinear elements using the same idea?

Constructing the Global Basis

Construct a basis on the element E_m from the reference basis $(\hat{\varphi}_j)_{j=1}^{N_p} : E_m \to \mathbb{R}$.

What's the gradient of this basis?

Assembling a Linear System

Express the matrix and vector elements in

$$\sum_{j=1}^{N_p} u_j a(\varphi_j, \varphi_i) = g(\varphi_i) \quad \text{for } i = 1, \ldots, N_p.$$

Integrals on the Reference Element

Evaluate

$$\int_E \kappa(\boldsymbol{x}) \nabla_{\boldsymbol{x}} \varphi_i(\boldsymbol{x})^T \nabla_{\boldsymbol{x}} \varphi_j(\boldsymbol{x}) d\boldsymbol{x}.$$

And now the RHS functional.

Inhomogeneous Dirichlet BCs

Handle an inhomogeneous boundary condition $u(\boldsymbol{x}) = \eta(\boldsymbol{x})$ on $\partial\Omega$.

Demo

- **Demo:** Developing FEM in 2D
- **Demo:** 2D FEM Using Firedrake
- **Demo:** Rates of Convergence

Outline

Conditions on the Mesh

Let Ω be a polygonal domain.

Admissibility (Braess, Def. II.5.1)

A partition (mesh) $\mathcal{T} = \{E_1, \ldots, E_M\}$ of Ω into triangular or quadrilateral elements is called admissible if

Give an example of a non-admissible partition.

Mesh Resolution, Shape Regularity

Definition (Diameter)

Mesh Resolution

Definition (Shape Regularity (Braess, Def. II.5.1))

A family of partitions $\{\mathcal{T}_h\}$ is called shape regular if

Cone Conditions

Definition (Lipschitz Domain)

A bounded domain $\Omega \subset \mathbb{R}^n$ is called a Lipschitz domain provided that...

Lipschitz domains satisfy a cone condition:

Theorem (Rellich Selection Theorem (Braess, Thm. II.1.9))

Let $m \geq 0$, let Ω be Lipschitz. Then the imbedding $H^{m+1}(\Omega) \to H^m(\Omega)$ is compact, i.e. any bounded sequence in the range of the imbedding has a

The Interpolation Operator

Theorem (Interpolation Operator (Braess, Lemma II.6.2))

Let $\Omega \subset \mathbb{R}^2$ be Lipschitz. Let $t \geq 2$, and $z_1, z_2, \ldots, z_s$ are $s := t(t+1)/2$ prescribed points in $\bar{\Omega}$ such that the interpolation operator $I : H^t \to \mathbb{P}^{t-1}$ is well-defined. Then there exists a constant c so that for $u \in H^t(\Omega)$

Theorem (Approx. for Congruent $\triangle$ (Braess, Remark II.6.5))

Let $E_h := h\hat{E}$, i.e. a scaled version of a reference triangle, with $h \leq 1$. Then, for $0 \leq m \leq t$, there exists a C so that

Approximation for Congruent Triangles: Proof (1/2)

Set up a function on E_h and $\hat{E}$. Work out the scaling for the derivative.

Work out the scaling for the Sobolev seminorm.

Work out the scaling for the Sobolev norm. Recall $h \leq 1$.

Approximation for Congruent Triangles: Proof (1/2)

$$\|u - Iu\|_{H^m(E_h)} \le Ch^{t-m} |u|_{H^t(E_h)} \quad (0 \le m \le t)$$

- $|v|^2_{H^\ell(\hat{E})} = |u|^2_{H^\ell(E_h)}$
- $\|u\|^2_{H^m(E_h)} \le C' h^{-2m+2} \|v\|^2_{H^m(\hat{E})}$

Prove the estimate.

H^m Polynomial Approximation on Meshes

Definition (Broken Norm)

Given a partition $\mathcal{T}_h = \{E_i\}_{i=1}^{M}$ and a function u such that $u \in H^m(E_i)$,

Approximation Theorem (Braess, Theorem II.6.4)

Let $t \geq 2$, suppose $\mathcal{T}_h$ is a shape-regular triangulation of Ω. Then there exists a constant c such that, for $0 \leq m \leq t$ and $u \in H^t(\Omega)$,

Outline

Weak Forms as Minimization Problems

Let V be a linear space, and $a : V \times V \to \mathbb{R}$ a bilinear form, and $g \in V'$.

Theorem (Solutions of Weak Forms are Quadratic Form Minimizers)

If a is SPD, then

attains its minimum over V at u iff $a(u, v) = g(v)$ for all $v \in V$.

Example: Lagrange Multipliers in $\mathbb{R}^2$

$$\begin{aligned} f(x,y) = x^2 + y^2 &\quad \rightarrow \quad \text{min!} \\ g(x,y) = x + y &\quad = \quad 2 \end{aligned}$$

Write down the Lagrangian.

Write down a necessary condition for a constrained minimum.

Saddle Point Problems

X, M Hilbert spaces. $a : X \times X \to \mathbb{R}$ and $b : X \times M \to \mathbb{R}$ continuous bilinear forms, $f \in X'$, $g \in M'$. Minimize

$$J(u) = \frac{1}{2}a(u,u) - \langle f, u\rangle \qquad \text{subject to} \qquad b(u,\mu) = \langle g, \mu\rangle \quad (\mu \in M).$$

Apply the method of the Lagrange multipliers.

Example: Saddle Point Problem in $\mathbb{R}^2$

$$f(x,y) = x^2 + y^2 \quad \rightarrow \quad \text{min!}$$
$$g(x,y) = x + y \quad = \quad 2$$

Lagrangian: $\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda g(x,y) = x^2 + y^2 + \lambda(x + y - 2)$.

Show that $x = y = 1$, $\lambda = -2$ is a saddle point.

Stokes Equation

$$\begin{aligned} \Delta \boldsymbol{u} + \nabla p &= -\boldsymbol{f} \quad (x \in \Omega), \\ \nabla \cdot \boldsymbol{u} &= 0 \quad (x \in \Omega), \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad (x \in \partial\Omega). \end{aligned}$$

What are the pieces?

Stokes: Properties

$$\begin{aligned} \Delta \boldsymbol{u} + \nabla p &= -\boldsymbol{f} \quad (x \in \Omega), \\ \nabla \cdot \boldsymbol{u} &= 0 \quad (x \in \Omega), \\ \boldsymbol{u} &= \boldsymbol{u}_0 \quad (x \in \partial\Omega). \end{aligned}$$

Can we choose any $\boldsymbol{u}_0$?

Does Stokes fully determine the pressure?

Stokes: Variational Formulation

$$\Delta \boldsymbol{u} + \nabla p = -\boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{u} = 0 \quad (x \in \partial\Omega).$$

Choose some function spaces (for homogeneous $\boldsymbol{u}_0 = 0$).

Derive a weak form.

Solvability of Saddle Point Problems

The Stokes weak form is clearly in saddle-point form.
Do all saddle point problems have unique solutions?

The inf-sup Condition

$$\begin{aligned} a(u,v) + b(v,\lambda) &= \langle f, v\rangle \quad (v \in X), \\ b(u,\mu) &= \langle g, \mu\rangle \quad (\mu \in M). \end{aligned}$$

Theorem (Brezzi's splitting theorem (Braess, III.4.3))

The saddle point problem has a unique solution if and only if

- *The bilinear form $a(\cdot,\cdot)$ is V-elliptic, where $V = \{u : b(u,\mu) = 0 \text{for all } \mu \in M\}$, i.e. there exists $c_0 > 0$ so that*

- *There exists a constant $c_2 > 0$ so that (inf-sup or LBB condition):*

Interpreting the inf-sup Condition

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} = M \begin{bmatrix} A & \\ & -BA^{-1}B^T \end{bmatrix} M^T$$

$$a(v,v) \geq c_0 \left\| v \right\|_X^2, \qquad \inf_{\mu \in M} \sup_{v \in X} \frac{b(v,\mu)}{\left\| v \right\|_X \left\| \mu \right\|_M} \geq c_2.$$

For any given v, can we expect $b(v,\mu)$ to be nonzero for all μ?

What is the inf-sup condition saying?

Why does it suffice for a to be V-elliptic?

inf-sup and Stokes

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega J_{\boldsymbol{u}} : J_{\boldsymbol{v}}, \qquad \text{where } A : B = \operatorname{tr}(AB^T), \\ b(\boldsymbol{v}, q) &= \int_\Omega \nabla \cdot \boldsymbol{v} q. \end{aligned}$$

Find $(\boldsymbol{u}, p) \in X \times M$ so that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{L^2} \quad (\boldsymbol{v} \in X), \\ b(\boldsymbol{u}, q) &= 0 \quad (q \in M). \end{aligned}$$

Theorem (Existence and Uniqueness for Stokes (Braess, III.6.5))

There exists a unique solution of this system when $\boldsymbol{f} \in H^{-1}(\Omega)^n$.

(based on results due to Ladyšenskaya, Nečas)

Discretizations for Stokes

Demo: 2D Stokes Using Firedrake (P^1-P^1)

Give a heuristic reason why P^1-P^1 might not be great.

Demo: Bad Discretizations for 2D Stokes

Establishing a Discrete inf-sup Condition

Suppose $b : X \times M \to \mathbb{R}$ satisfies inf-sup. Subspaces $X_h \subseteq X$, $M_h \subseteq M$.

Fortin's Criterion ([Fortin 1977])

Suppose there exists a bounded projector $\Pi_h : X \to X_h$ so that

If $\|\Pi_h\| \leq c$ for some constant c independent of h, then b satisfies the inf-sup-condition on $X_h \times M_h$.

H^1-Boundedness of the L^2-Projector

Assume H^2-regularity and a uniform triangulations $\mathcal{T}_h$. (Not in general!)

H^1-Boundedness of the L^2-Projector (Braess Corollary II.7.8)

Let π_h^0 be the L_2-projector onto a finite element space $V_h \subset H^1(\Omega)$. Then, for an h-independent constant c,

Ingredients?

H^1-Boundedness of the L^2-Projector

Does H^1 boundedness of the H^1 projector hold?

How would this break down without the uniformity assumption?

Bubbles and the MINI Element

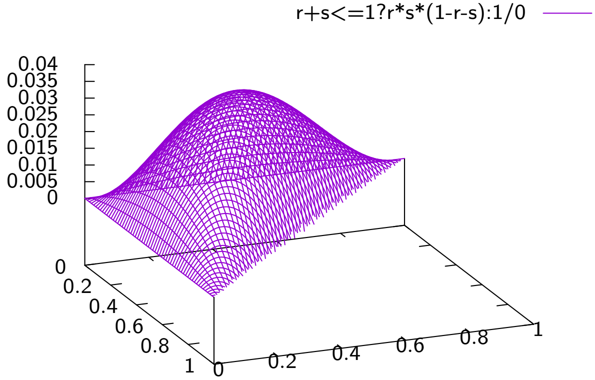
What is a bubble function?

Let B^3 be the span of the bubble function and $\mathcal{T}_h$ the triangulation.

Define the MINI variational space $X_h \times M_h$.

Computational impact of the bubble DOF?

The Bubble in Pictures



MINI Satisifies an inf-sup Condition (1/4)

MINI satisifes inf-sup (Braess Theorem III.7.2)

Assume Ω is convex or has a smooth boundary. Then the MINI variational space satisfies an inf-sup condition for every variational form that itself satisfies one.

MINI Satisifies an inf-sup Condition (2/4)

Create a projector onto the bubble space B^3.

What does this bubble projector do?

Do we have an estimate for the bubble projector?

MINI Satisifies an inf-sup Condition (3/4)

Make an overall projector Π_h onto X_h.

Show Fortin's criterion for Π_h.

MINI Satisifies an inf-sup Condition (4/4)

- $\left\|\pi_h^0 v\right\|_{H^1} \leq c_1 \|v\|_{H^1}$ for L^2 projector $\pi_h^0 : H_0^1 \to \mathcal{M}_h$.
- $\left\|v - \pi_h^0 v\right\|_{L^2} \leq c_2 h \, |v|_{H^1}$.
- $\left\|\pi_h^1 v\right\|_{L^2} \leq c_3 \|v\|_{L^2}$.

Show H^1-boundedness of Π_h.

Demo

Demo: 2D Stokes Using Firedrake (MINI and Taylor-Hood)

Outline

Lax-Milgram, General Case

Let V be Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

Theorem (Lax-Milgram, General Case)

Let a be a V-elliptic bilinear form, and let g be a bounded linear functional on V.
Then there exists a unique $u \in V$ so that $a(u, v) = g(v)$ for all $v \in V$.

Lax-Milgram Proof (2/5)

$a(u, v) = \langle v, Tu \rangle$. Show linearity of T.

Show boundedness $\Leftrightarrow$ continuity of T.

Lax-Milgram Proof (3/5)

$a(u, v) = \langle v, Tu \rangle$. Show that T has closed range. (Needed for Hilbert projection, which is needed for onto.)

Lax-Milgram Proof (4/5)

$a(u, v) = \langle v, Tu \rangle$. Show that T is onto V.

Lax-Milgram Proof (5/5)

Show existence of the solution u.

Show uniqueness of the solution u.

Outline

Outline

Conservation laws

Goal: Solve *conservation laws* on bounded domain $\Omega \subset \mathbb{R}^n$:

$$\boldsymbol{q}_t + \nabla \cdot \boldsymbol{F}(\boldsymbol{q}) = 0$$

Example: Maxwell's Equations

$$\begin{aligned} \partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} &= -\boldsymbol{j}, & \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} &= 0, \\ \nabla \cdot \boldsymbol{D} &= \rho, & \nabla \cdot \boldsymbol{B} &= 0. \end{aligned}$$

What do we do with the divergence constraints?

Rewriting Maxwell's

Let $\boldsymbol{q} = (D_x, D_y, D_z, B_x, B_y, B_z)^T$. Consider $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$.

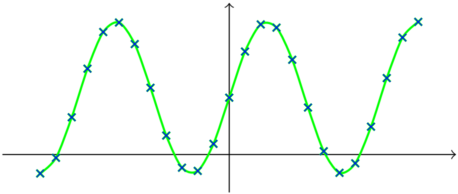
$$\partial_t \boldsymbol{D} - \nabla \times \boldsymbol{H} = -0, \qquad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0.$$

Rewrite in conservation law form: $\boldsymbol{q}_t + \nabla \cdot F(\boldsymbol{q}) = 0$

Could we also define $\boldsymbol{q} = (E_x, E_y, E_z, H_x, H_y, H_z)^T$?

Outline

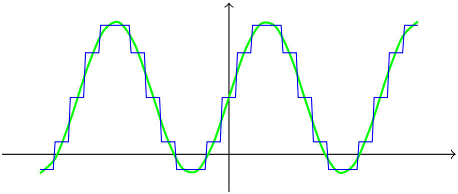
Solving $q_t + aq_x = 0$: Finite Differences



$$D_t^- + aD_x^- = 0$$

$$D_t^+ f := \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Solving $q_t + aq_x = 0$: Finite Volume

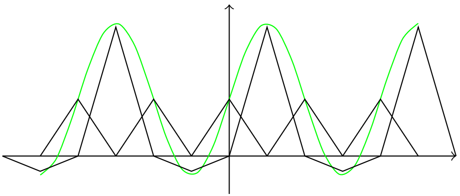


$$\bar{q}_k := \int_{(k-1/2)\Delta x}^{(k+1/2)\Delta x} q(x)dx$$

$$\Delta x \partial_t \bar{q}_k + f^{k+1/2} - f^{k-1/2} = 0$$

$f^{k\pm 1/2}$: flux “reconstructions”

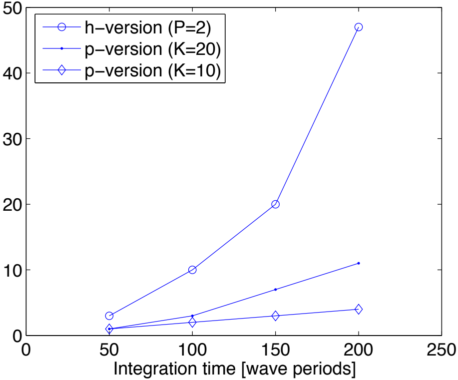
Solving $q_t + aq_x = 0$: Finite Elements



$$\int_\Omega q_t^N \phi + a q_x^N \phi dx = 0$$

for ϕ in a test space.

Do we really want high order?



Time to compute solution at 5% error

Big assumption?

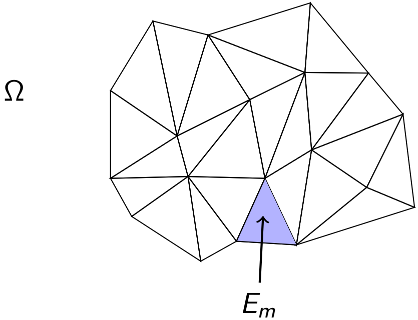
Figure from talk by Jan Hesthaven

Summarizing

Want flexibility of finite elements *without* the drawbacks.

Outline

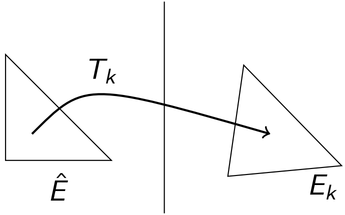
Developing the Scheme



What do do about unbounded domains?

Dealing with the Mesh, Part I

For each cell E_k, find a ref-to-global map T_k:



$$T_k : \hat{E} \to E_k$$

$$\boldsymbol{x} = (x, y, z) = T_k(r, s, t) = T_k(\boldsymbol{r})$$

- T_k affine for straight-sided simplices: $T_k(\boldsymbol{r}) = A\boldsymbol{r} + \boldsymbol{b}$
- Curved elements also possible: iso/sub/super-parametric

Dealing with the Mesh, Part II

Based on knowledge of how to do this on $\hat{E}$:

Can now *integrate* on Ω:

and *differentiate* on Ω:

Jacobian of T_k^{-1}?

Dealing with the Mesh, Part III

Approximation basis set on E_k?

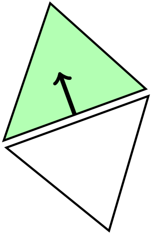
What function space do we get if Ψ is non-affine?

Going Galerkin

$$\int_{E_k} q_t^k \phi + (\nabla \cdot F^k)\phi dx = 0$$

Integrate by parts:

Problem?



Strong-Form DG

Weak form:

$$0 = \int_{E_k} q_t^k \phi dx - \int_{E_k} F^k \cdot \nabla \phi dx + \int_{\partial E_k} (F^k \cdot \hat{\boldsymbol{n}})^* \phi dx$$

Integrate by parts *again*:

Outline

Accuracy and Stabillity

In DG: what provides accuracy? what provides stability?

Following slides based on material by Tim Warburton

Stability: Basic Setup (1/2)

$$0 = \int_{E_k} q_t^k \phi dx - \int_{E_k} F^k \cdot \nabla \phi dx + \int_{\partial E_k} (F^k \cdot \hat{\boldsymbol{n}}) \phi dS_x$$

Stability: Basic Setup (2/2)

$$\frac{\partial_t \|q_k\|^2_{2,E_k}}{2} = \int_{E_k} a q_k \partial_x q_k \, dx - \int_{\partial E_k} (a q_k n_x)^* q_k \, dS_x$$

Stability: Going Global

$$\frac{\partial_t \|q_k\|^2_{2,E_k}}{2} = \int_{\partial E_k} \frac{a(q_k)^2 n_x}{2} - (a q_k n_x)^* q_k \, dS_x$$

Gather up

$$\frac{\partial_t \|q_k\|^2_{2,\Omega}}{2} = \sum_{f \in \text{faces}} \Big(\int_f \frac{a(q_k^+)^2 n_x^+}{2} - (a q_k n_x)^*_+ q_k^+ \, dS_x \\ + \int_f \frac{a(q_k^-)^2 n_x^-}{2} - (a q_k n_x)^*_- q_k^- \, dS_x \Big)$$

Picking a Flux

Want:

$$(*) = \left(a n_x^- \frac{q_k^- + q_k^+}{2} - (a q_k n_x)^*_- \right) (q_k^- - q_k^+) \overset{!}{\leq} 0$$

Ideas?

Picking a flux, attempt two

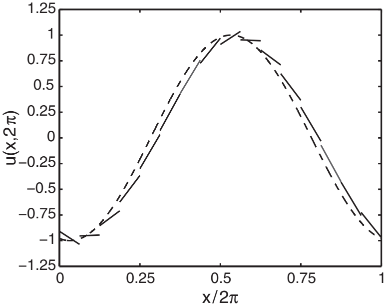
Want:

$$(*) = \left(an_x^- \frac{q_k^- + q_k^+}{2} - (aq_k n_x)^*_-\right)(q_k^- - q_k^+) \overset{!}{\leq} 0$$

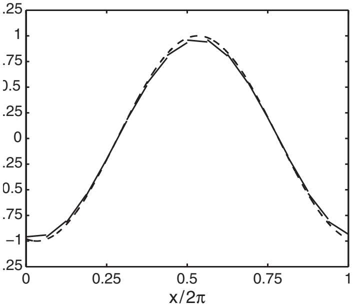
More ideas?

Comparing Fluxes (1/3)

Central



Upwind



Upwind penalizes jumps!

Figure from talk by Jan Hesthaven

Comparing Fluxes (2/3)

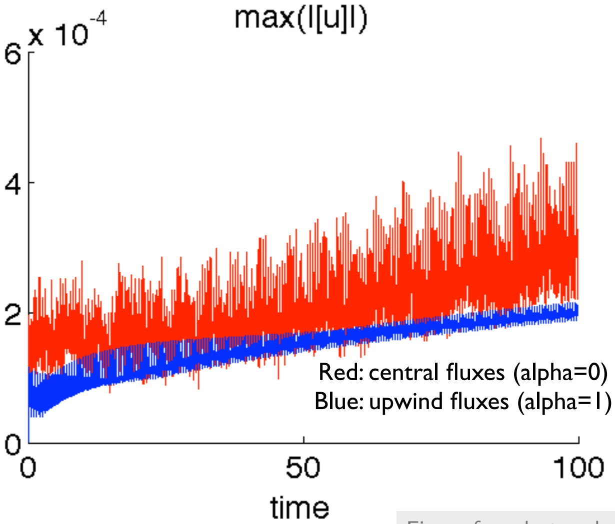


Figure from lecture by Tim Warburton

Comparing Fluxes (3/3)

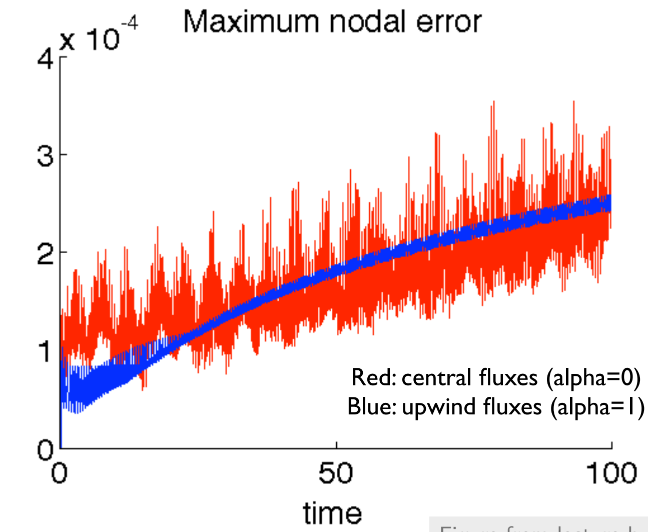


Figure from lecture by Tim Warburton

Stability Analysis

Clif notes on flux choice?

Swept under the rug: Boundary conditions

Element coupling (and BCs) done *weakly*

Accuracy

Stability: (preliminary version) done!

Accuracy: Depends on approximation properties!

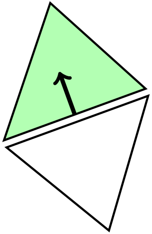
Systems of Conservation Laws

What to do about systems?

What about multiple dimensions?

We've dealt with 1D systems.

How about the move to multiple dimensions?



Simultaneous Diagonalization

2D second-order wave equation across a boundary with normal n:

Demo: Finding Numerical Fluxes for DG (Part 1)

Jumps and Averages

Jump and average of a scalar quantity:

Jump and average of a vector quantity:

A Flux for Maxwell's

Wanted to solve Maxwell's equation in the time domain. Numerical flux?

Either look in the literature:

$$\hat{\boldsymbol{n}} \cdot (\boldsymbol{F}_N - \boldsymbol{F}_N^*) := \frac{1}{2} \begin{pmatrix} \{Z\}^{-1} \hat{\boldsymbol{n}} \times (Z^+ [\![\boldsymbol{H}]\!] - \alpha \hat{\boldsymbol{n}} \times [\![\boldsymbol{E}]\!]) \\ \{Y\}^{-1} \hat{\boldsymbol{n}} \times (-Y^+ [\![\boldsymbol{E}]\!] - \alpha \hat{\boldsymbol{n}} \times [\![\boldsymbol{H}]\!]) \end{pmatrix}.$$

or derive yourself: **Demo:** Finding Numerical Fluxes for DG (Part 2)

Good news: Scheme mathematically complete.

Outline

Implementing DG

Weak form:

$$0 = \int_{E_k} q_t^k \phi dx - \int_{E_k} F^k \cdot \nabla \phi dx + \int_{\partial E_k} (F^k \cdot n)^* \phi dx$$

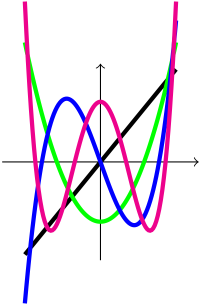
What do the DoFs mean?

Modes

Function spaces same as for FEM: P^N, Q^N.

Numerically: better to use orthogonal polynomials with

$$\int_{\hat{E}} \phi_i \phi_j = \delta_{i,j}$$



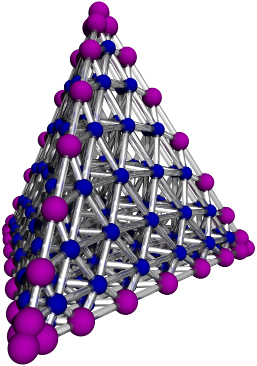
Nodes

Define set of interpolation nodes $(\xi_i)_{i=1}^{N_p}$ and ℓ_i their Lagrange basis.

Define *generalized Vandermonde matrix*

$$V_{ij} := \phi_j(\xi_i)$$

ξ_i determine $\operatorname{cond}(V)$!



In Matrix Form

$$0 = \int_{E_k} q_t^k \phi dx - \int_{E_k} F^k \cdot \nabla \phi dx + \int_{\partial E_k} (F^k \cdot n)^* \phi dx$$

Write in matrix form:

Explicit Time Integration

$$0 = \mathcal{M}^k \partial_t u^k - \sum_\nu \mathcal{S}^{k,\partial_\nu}[F(u^k)] + \sum_{A \subset \partial E_k} \mathcal{M}^{k,A}(\hat{n} \cdot F)^*$$

How can we do time integration on this weak form?

Trick: Multiple face mass matrices

Applying multiple face mass matrices at once:

DG and Modern Computers: Possible Advantages

DG on modern processor architectures: Why?