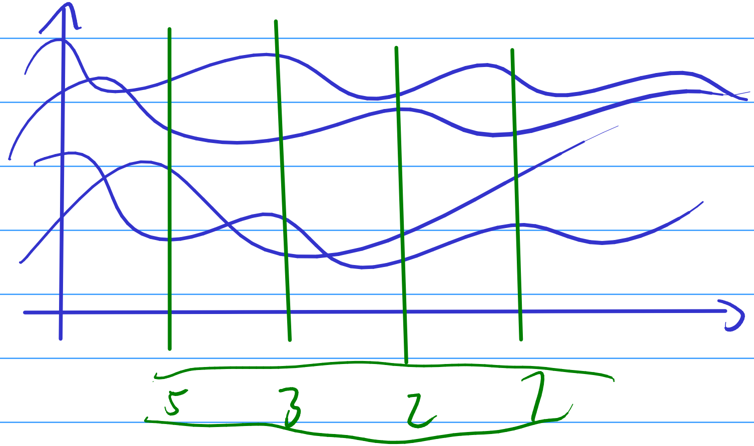
Compact sets

Definition 11. (Precompact/Relatively compact) *$M \subseteq X$ precompact:* $\Leftrightarrow$ *all sequences $(x_k) \subset M$ contain a subsequence converging in X*

Definition 12. (Compact/'Sequentially complete') *$M \subseteq X$ compact:* $\Leftrightarrow$ *all sequences $(x_k) \subset M$ contain a subsequence converging in M*

- Precompact $\Rightarrow$ bounded
- Precompact $\Leftrightarrow$ bounded (finite dim. only!)

◦ Counterexample?


5
3
2
1

Compact Operators

X, Y: Banach spaces

Definition 13. (Compact operator) *$T\colon X \to Y$ is compact $:\Leftrightarrow$ T(bounded set) is precompact.*

- T, S compact $\Rightarrow \alpha T + \beta S$ compact
- One of T, S compact $\Rightarrow S \circ T$ compact
- T_n all compact, $T_n \to T$ in operator norm $\Rightarrow T$ compact

Questions:

- Let $\dim T(X) < \infty$. Is T compact?
- Is the identity operator compact?

Intuition about Compact Operators

- Compact operator: As finite-dimensional as you're going to get in infinite dimensions.
- Not clear yet–but they are moral (∞-dim) equivalent of a matrix having low numerical rank.
- Are compact operators continuous (=bounded)?
- What do they do to high-frequency data?
- What do they do to low-frequency data?

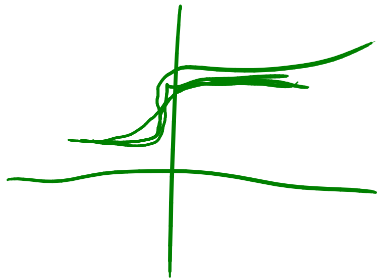
Arzelà-Ascoli

Let $G \subset \mathbb{R}^n$ be compact.

Theorem 14. (Arzelà-Ascoli) *$U \subset C(G)$ is precompact iff it is bounded and equicontinuous.*

Equicontinuous means
For all $x, y \in G$
for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in U$
if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Continuous means:
For all $x, y \in G$
for all $\epsilon > 0$ there exists a $\delta > 0$ such that
if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.



- Intuition?

- “Uniformly continuous”?
- When does *uniform continuity* happen?

continuous f. on a closed domain [——]

6.3 Integral Operators

Integral Operators are Compact

Theorem 15. (Continuous kernel ⇒ compact [Kress LIE Thm. 2.21]) $G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. *Then*

↳ closed

$$(A\phi)(x) := \int_G K(x,y)\,\phi(y)\,dy.$$

is compact on $C(G)$.

∗ Use A-A. (a statement about compact sets)
What is there to show?
- ○ Pick $U \subset C(G)$. $A(U)$ bounded? ✓
- ○ $A(U)$ equicontinuous?

$U \subset C(G)$.
$u \in U$: $\|u\|_\infty \le c$
$\|A(u)\|_\infty \le \|A\|\,\|\varphi\|_\infty$

$\forall x, x' \in G$: $\forall \varepsilon > 0\ \exists \delta > 0$; for all $\varphi \in A(U)$; ⇝ $\varphi = Au$

$|x - x'| < \delta \Rightarrow |\varphi(x) - \varphi(x')| < \varepsilon$

$|\varphi(x) - \varphi(x')| = |\int_G k(x,y)\,\varphi(y)\,dy - \int_G k(x',y)\,u(y)\,dy|$

$$|\varphi(x) - \varphi(x')| = \left| \int_G k(x,y)\, u(y)\, dy - \int_G k(x',y)\, u(y)\, dy \right|$$

$$\le \int_G |k(x,y) - k(x',y)|\, \underbrace{|u(y)|}_{\le \|u\|_\infty}\, dy$$

$$\le \|u\|_\infty \int_G |k(x,y) - k(x',y)|\, dy < \varepsilon \qquad |x - x'| < \delta$$

"uniformly continuous"

From UC: $\exists$ a δ so that $|k(x,y) - k(x',y)| < \dfrac{\varepsilon}{\|u\|_\infty \int_G 1\, dx}$

Weakly singular

$G \subset \mathbb{R}^n$ compact

$\alpha \in (0, 2]$

$\alpha = 0 \quad |x-y|^{0-2}$

$\alpha = 2: \quad |x-y|^{2-2} = 1$

Definition 16. (Weakly singular kernel)

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n]$ such that*

$$|K(x, y)| \leq C \, |x - y|^{\alpha - n} \qquad (x \neq y)$$

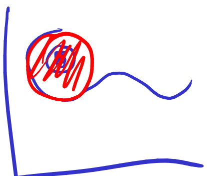
Theorem 17. (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.22])
K weakly singular. Then

$$(A\phi)(x) := \int_G K(x, y) \, \phi(y) \, dy.$$

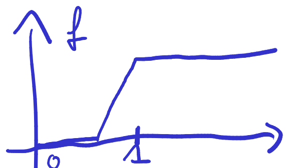
is compact on $C(G)$.

- Outline the proof.

D'S



$$\tilde{K}_n(x, y) = f(n\|x-y\|) \, K(x, y)$$



$$A\varphi(x) = \int_G k(x,y)\,\varphi(y)\,dy \qquad |k(x,y)| \le |x-y|^{-1.999}$$

$$\|\varphi\|_\infty \le C$$

$$|A\varphi(x)| \le \int_G |k(x,y)\,\varphi(y)|\,dy$$

$$\le \|\varphi\|_\infty \int_G |k(x,y)|\,dy$$

$$\le \|\varphi\|_\infty \int_G |x-y|^{-1.999}\,dy \qquad \tilde{y} = x-y$$

$$\le \|\varphi\|_\infty \int_G \|\tilde{y}\|^{-1.999}\,d\tilde{y}$$

$$\le C\|\varphi\|_\infty \int_0^R r^{-1.999}\,r\,dr$$

$$\le C\|\varphi\|_\infty \underbrace{\int_0^R r^{-0.999}\,dr}$$

Weakly singular (on surfaces)

- $\Omega \subset \mathbb{R}^n$ bounded, open, C^1

2D; (not ok) $\alpha = 0$ $r^{0-2+1} = \frac{1}{r}$; $\alpha = 1$ $r^{1-2+1} = r^0$

Definition 18. (Weakly singular kernel (on a surface))

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n-1]$ such that*

$$|K(x, y)| \leq C\, |x - y|^{\alpha - n + 1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Theorem 19. (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.23]) *K weakly singular on $\partial\Omega$. Then*

$$(A\phi)(x) := \int_G K(x, y)\, \phi(y)\, dy.$$

is compact on $C(G)$.

6.4 Riesz and Fredholm

Riesz Theory (I)

- Still trying to solve

$$L\phi := (I - A)\phi = \phi - A\phi = f$$

with A compact.

> **Theorem 20. (First Riesz Theorem** [Kress, Thm. 3.1]**)** *$N(L)$ is finite-dimensional.*

Questions:

- What is $N(L)$ again?
- Why is this good news?
- Show it.

$\rightsquigarrow \quad \varphi \in N(L) \iff (I - A)\varphi = 0$

$\iff \quad \varphi - A\varphi = 0$

$\iff \quad \varphi = A\varphi$

$A\varphi(x) = \int k(x,y)\varphi(y)\,dy$

Riesz Theory (Part II)

Theorem 21. **(Riesz theory** [Kress, Thm. 3.4]**)** *A compact.* ***Then:***

- $(I-A)$ *injective* (1-to-1) $\Leftrightarrow$ $(I-A)$ *surjective* (onto)
  - *It's either bijective or neither s nor i.* (hom. unique)
- *If $(I-A)$ is bijective, $(I-A)^{-1}$ is bounded.*

- Rephrase for solvability
- Main impact?
- Key shortcoming?

$I-A$ onto:

range $(I-A) = C(G)$

$\varphi(x) - \int K(x,y)\varphi(y)\,dy = f(x)$

$I-A$ one-to-one:

$(I-A)\varphi = \psi = (I-A)\varphi'$

$\Leftrightarrow [(I-A)\varphi = 0 \Rightarrow \varphi = 0]$

$\Leftrightarrow N(I-A) = \{0\}$

$\Leftrightarrow [\varphi - \int_G k(x,y)\varphi(y)\,dy = 0 \Rightarrow \varphi = 0]$

Hilbert spaces

Hilbert space: Banach space with a norm coming from an inner product:

$$(\alpha x + \beta y, z) = ?$$

$$(x, \alpha y + \beta z) = ?$$

$$(x, x) ?$$

$$(y, x) = ?$$

- Is $C^0(G)$ a Hilbert space?
- Name a Hilbert space of functions.
- Is $C^0(G)$ "equivalent" to $L^2(G)$?
- Why do compactness results transfer over nonetheless? Hint: What is

Adjoint Operators

Definition 22. (Adjoint oeprator) *A^* called adjoint to A if*

$$\rightarrow (Ax, y) = (x, A^* y)$$

for all x, y.

Facts: $K(x,y)$

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

◦ What is the adjoint operator in finite dimensions? (in matrix representation)

- What do you expect to happen with integral operators?
- Adjoint of the single-layer?
- Adjoint of the double-layer?

Fredholm Alternative

Theorem 23. (Fredholm Alternative [Kress LIE Thm. 4.14])
$A: X \to X$ compact. ***Then either:***

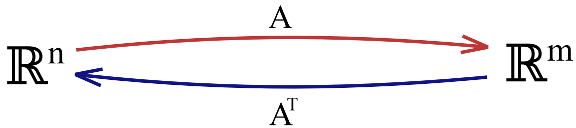
- *$I - A$ and $I - A^*$ are bijective*

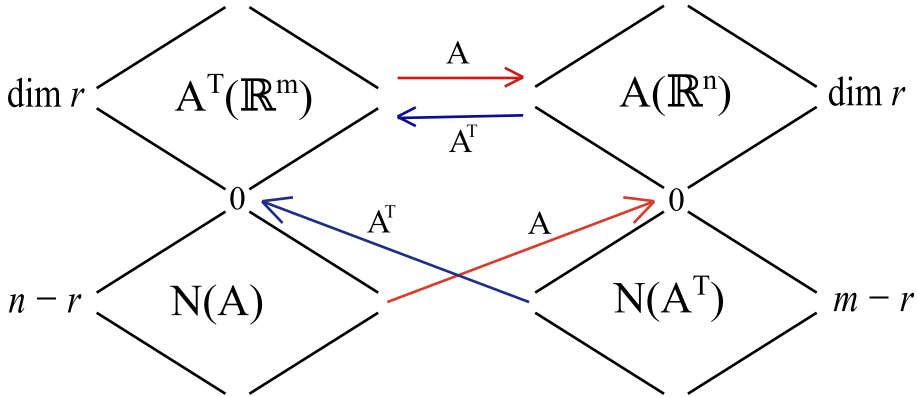
or:

- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^\perp$
- $(I - A^*)(X) = N(I - A)^\perp$

- Seen these statements before?
- Translate to language of integral equation solvability:
- What about symmetric kernels ($K(x, y) = K(y, x)$)?
- Where to get uniqueness?

Fundamental Theorem of Linear Algebra





6.5 A Tiny Bit of Spectral Theory

Spectral Theory: Terminology

$A\colon X \to X$ bounded, λ is a _ _ _ _ value:

Definition 24. (Eigenvalue) *There exists an element* $\phi \in X$, $\phi \neq 0$ *with* $A\phi = \lambda\phi$.

Definition 25. (Regular value) *The "resolvent"* $(\lambda I - A)^{-1}$ *exists and is bounded.*

- Can a value be regular and "eigen" at the same time?
- What's special about ∞-dim here?

Definition 26. (Resolvent set) $\rho(A) := \{\lambda \text{ is regular}\}$

Definition 27. (Spectrum) $\sigma(A) := \mathbb{C} \setminus \rho(A)$

Spectral Theory of Compact Operators

Theorem 28. *$A\colon X \to X$ compact linear operator, X ∞-dim.*
Then:

- $0 \in \sigma(A)$ *(show!)*
- $\sigma(A) \setminus \{0\}$ *consists only of eigenvalues*
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0*

- Show first part.
- Show second part.
- Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?
- **Recap:** What do compact operators do to high-frequency data?
- Don't confuse $I - A$ with A itself!

7 Singular Integrals and Potential Theory

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\,\sigma(y)\,ds_y$$

$$(S'\sigma)(x) := PV\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\,\sigma(y)\,ds_y$$

$$(D\sigma)(x) := PV \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y$$

Definition 29. (Harmonic function) $\triangle u = 0$

- Where are layer potentials harmonic?

On the double layer again

Is the double layer *actually* weakly singular?

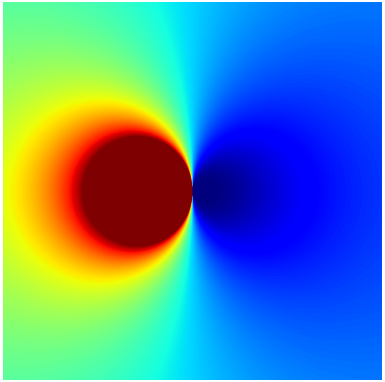
Recap:

Definition 30. (Weakly singular kernel)

- *K defined, continuous everywhere except at* $x = y$
- *There exist* $C > 0$, $\alpha \in (0, n-1]$ *such that*

$$|K(x,y)| \leq C\, |x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

$$\frac{\partial}{\partial_x}\log(|0 - x|) = \frac{x}{x^2 + y^2}$$



- Singularity with approach on $y = 0$?
- Singularity with approach on $x = 0$?

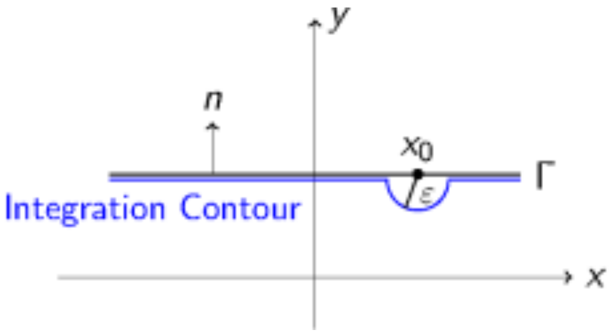
So life is simultaneously worse and better than discussed.
How about 3D? $(-x/|x|^3)$

Would like an analytical tool that requires 'less' fanciness.

Cauchy Principal Value

- But I don't **want** to integrate across a singularity!
- $$\int_{-1}^{1} \frac{1}{x}\, dx?$$

Principal Value in n dimensions



- Again: Symmetry matters!
- What about even worse singularities?

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\,\sigma(y)\,ds_y$$

$$(S'\sigma)(x) := \mathrm{PV}\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\,\sigma(y)\,ds_y$$

$$(D\sigma)(x) := \mathrm{PV} \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y$$

Important for us: Recover ‘average’ of interior and exterior limit without having to refer to off-surface values.

Green's Theorem

Theorem 31. (Green's Theorem [Kress LIE Thm 6.3])

$$\int_{\Omega} u \triangle v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u \, (\hat{n} \cdot \nabla v) \, ds$$

$$\int_{\Omega} u \triangle v - v \triangle u = \int_{\partial\Omega} u \, (\hat{n} \cdot \nabla v) - v \, (\hat{n} \cdot \nabla u) \, ds$$

- If $\triangle v = 0$, then

$$\int_{\partial\Omega} \hat{n} \cdot \nabla v = ?$$

- What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

Green's Formula

Theorem 32. **(Green's Formula** [Kress LIE Thm 6.5]**)** *If* $\triangle u = 0$, *then*

$$(S(\hat{n}\cdot\nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

- Suppose I know 'Cauchy data' $(u|_{\partial D}, \hat{n}\cdot\nabla u|_{\partial D})$ of u. What can I do?
- What if D is an exterior domain?

Things harmonic functions (don't) do

Theorem 33. (Mean Value Theorem [Kress LIE Thm 6.7]**)** *If* $\Delta u = 0$,

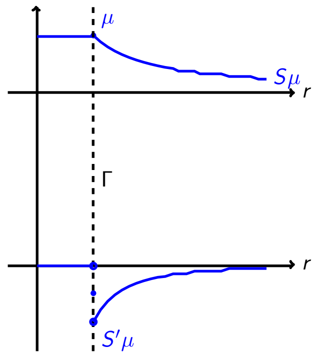
$$u(x) = \fint_{B(x,r)} u(y)\,dy = \fint_{\partial B(x,r)} u(y)\,dy$$

- Define $\fint$?
- Trace back to Green's Formula (say, in 2D):

Theorem 34. (Maximum Principle [Kress LIE 6.9]**)** *If* $\triangle u = 0$ *on compact set* $\bar{D}$*:*
u attains its maximum on the boundary.

- Suppose it were to attain its maximum somewhere inside an open set...
- What do our constructed harmonic functions (i.e. layer potentials) do there?

Jump relations



Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".)

[Kress LIE Thm. 6.14, 6.17,6.18]

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I\right)(\sigma)(x_0) \quad \Rightarrow \quad [D\sigma] = \sigma$$

$$[\not{D}'\sigma] = 0$$

- Truth in advertising: Assumptions on Γ?

Green's Formula at Infinity

TODO: Needed?

$\Omega \subseteq R^n$ bounded, C^1, connected boundary, $\triangle u = 0$, u bounded

$$(S_{\partial\Omega}(\hat{n}\cdot\nabla u) - D_{\partial\Omega} u)(x) + (S_{\partial B_r}(\hat{n}\cdot\nabla u) - D_{\partial B_r} u)(x) = u(x)$$

for x between $\partial\Omega$ and B_r.

Now $r \to \infty$.

Behavior of individual terms?

Theorem 35. (Green's Formula in the exterior [Kress LIE Thm 6.10])

$$(S_{\partial\Omega}(\hat{n}\cdot\nabla u) - D_{\partial\Omega} u)(x) + \text{PV}\, u_\infty = u(x)$$

for some constant u_∞. Only for $n = 2$,

$$u_\infty = \frac{1}{2\pi r}\int_{|y|=r} u(y)\,ds_y.$$

Theorem 36. (Green's Formula in the exterior [Kress LIE Thm 6.10])

$$(S_{\partial\Omega}(\hat{n}\cdot\nabla u) - D_{\partial\Omega} u)(x) + u_\infty = u(x)$$

- Realize the power of this statement:
- Behavior of the fundamental solution as $r \to \infty$?
- How about its derivatives?

8 Boundary Value Problems

Boundary Value Problems: Overview

	Dirichlet	Neumann
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ ⊕ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ ⊙ may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert \to \infty$ ⊕ unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert \to \infty$ ⊕ unique

with $g \in C(\partial\Omega)$.

- What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)
- Dirichlet uniqueness: why?
- Neumann uniqueness: why?
- Truth in advertising: Missing assumptions on Ω?
- What's a DtN map?

Next mission: Find IE representations for each.

Uniqueness of Integral Equation Solutions

Theorem 37. (Nullspaces [Kress LIE Thm 6.20]**)**

- $N(I/2 - D) = N(I/2 - S') = \{0\}$
- $N(I/2 + D) = \mathrm{span}\{1\}$, $N(I/2 + S') = \mathrm{span}\{\psi\}$, where $\int \psi \neq 0$.

What to show?
Start with $I/2 - D$. What BVP?
Hint: Use jump relations, use BVP uniqueness, use both kinds of boundary data.
Next: $I/2 + D$ on exterior.
Show ϕ constant. Nullspace identified?
Extra conditions on RHS?
$(I - A)(X) = N(I - A^*)^{\perp}$
$\to$ "Clean" Existence for 3 out of 4.

Patching up Exterior Dirichlet

Problem: $N(I/2+S') = \{\psi\}$...but we do not know ψ.
Use a different kernel:

$$\hat{n}\cdot\nabla_y G(x,y) \qquad \to \qquad \hat{n}\cdot\nabla_y G(x,y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in\Omega$)

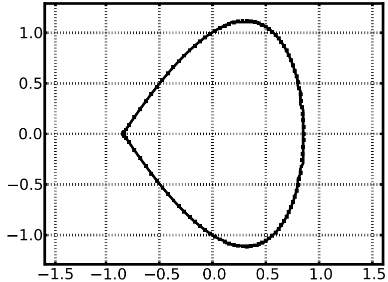
- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u=0$ on exterior.
- $|x|^{n-2}u(x) = ?$ on exterior
- Thus $\int \phi = 0$. Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2+D) = ?$

- Existence/uniqueness?

→ Existence for 4 out of 4.

○ Remaining key shortcoming of IE theory for BVPs?

Domains with Corners



What's the problem? (Hint: Jump condition for constant density)

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x, y)\, \phi(y)\, ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on} \pm \text{side} \rangle}{\pi} \phi$$

$\to$ non-continuous behavior of potential on Γ at x_0

What space have we been living in?

Fixes:

- I + Bounded (Neumann) + Compact (Fredholm)

- Use L^2 theory
  (point behavior "invisible")

Numerically: Needs consideration, but ultimately easy to fix.

(to end lec15)

9 Back from Infinity: Discretization

10 Computing Integrals: Approaches to Quadrature

11 Going General: More PDEs