Integral Operators are Compact

Theorem 15. (Continuous kernel ⇒ compact [Kress LIE Thm. 2.21]) *$G \subset \mathbb{R}^m$ compact, $K \in C(G^2)$. Then*

$$(A\phi)(x) := \int_G K(x, y)\,\phi(y)\,dy.$$

is compact on $C(G)$.

- Use A-A. (a statement about compact sets)
  What is there to show?
  - Pick $U \subset C(G)$. $A(U)$ bounded?
  - $A(U)$ equicontinuous?

Weakly singular

$G \subset \mathbb{R}^n$ compact

Definition 16. (Weakly singular kernel)

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n]$ such that*

$$|K(x, y)| \leq C\, |x - y|^{\alpha - n} \qquad (x \neq y)$$

Theorem 17. (Weakly singular kernel $\Rightarrow$ compact [Kress LIE Thm. 2.22])

K weakly singular. Then

$$(A\,\phi)(x) := \int_G K(x, y)\, \phi(y)\, dy.$$

is compact on $C(G)$.

- ○ Outline the proof.

Weakly singular (on surfaces)

○ $\Omega \subset \mathbb{R}^n$ bounded, open, C^1

Definition 18. (Weakly singular kernel (on a surface))

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n-1]$ such that*

$$|K(x,y)| \leq C\, |x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

Theorem 19. (Weakly singular kernel ⇒ compact [Kress LIE Thm. 2.23]) *K weakly singular on $\partial\Omega$. Then*

$$(A\phi)(x) := \int_G K(x,y)\,\phi(y)\,dy.$$

is compact on $C(G)$.

6.4 Riesz and Fredholm

Riesz Theory (I)

- Still trying to solve

$$L\,\phi := (I - A)\,\phi = \phi - A\,\phi = f$$

with A compact.

Theorem 20. (First Riesz Theorem [Kress, Thm. 3.1]) *$N(L)$ is finite-dimensional.*

Questions:

- What is $N(L)$ again?
- Why is this good news?
- Show it.

Riesz Theory (Part II)

Theorem 21. **(Riesz theory** [Kress, Thm. 3.4]**)** *A compact.* ***Then:***

- *$(I - A)$ injective $\Leftrightarrow$ $(I - A)$ surjective*
  (I) (II)
  - *It's either bijective or neither s nor i.*
- *If $(I - A)$ is bijective, $(I - A)^{-1}$ is bounded.*

○ Rephrase for solvability

○ Main impact?

○ Key shortcoming?

(I) $\varphi - \int k \varphi = 0 \Rightarrow \varphi = 0$

(II) $\forall f \in C: \varphi - \int k \varphi = f$

Hilbert spaces

Hilbert space: Banach space with a norm coming from an inner product:

$$(\alpha x + \beta y, z) = ?$$

$$(f,g) = \int f(x)\,\overline{g(x)}\,dx$$

$$(x, \alpha y + \beta z) = ?$$

$$(\vec{x}, \vec{y}) = \sum_i x_i y_i$$

$$\|f\|_\infty = \sup_{x \in G} |f(x)|$$

$$(x, x) ? \; 0 \; (=) \quad x = 0$$

$$(y, x) = ?$$

$$\|f\|_2 = \sqrt{(f,f)}$$

L^2

- ○ Is $C^0(G)$ a Hilbert space?
- ○ Name a Hilbert space of functions. →
- ○ Is $C^0(G)$ "equivalent" to $L^2(G)$?
- ○ Why do compactness results transfer over nonetheless? Hint: What is

$$|(f,g)| \leq \|f\|_2 \|g\|_2$$

Adjoint Operators

Definition 22. (Adjoint oeprator) *A^* called adjoint to A if*

$$(Ax, y) = (x, A^* y)$$

for all x, y.

Facts:

- A^* unique
- A^* exists
- A^* linear
- A bounded $\Rightarrow$ A^* bounded
- A compact $\Rightarrow$ A^* compact

- ◦ What is the adjoint operator in finite dimensions? (in matrix representation)

- What do you expect to happen with integral operators?
- Adjoint of the single-layer?
- Adjoint of the double-layer?

$$Af(x) = \int k(x,y)\, f(y)\,dy$$

$$(Af, g) = \int \int k(x,y)\, f(y)\,dy\; g(x)\,dx$$

$$= \int \int k(x,y)\; g(x)\,dx\; f(y)\,dy$$

$$K^*(x,y) = K(y,x)$$

$$= \int \int k(x,y)\; g(x)\,dx\; f(y)\,dy$$

$$= \int \underbrace{\int K^*(y,x)\; g(x)\,dx}_{A^*g}\; f(y)\,dy$$

$$= (f, A^*g)$$

$$S\varphi(x) = \int_\Gamma \log(|x-y|)\,\varphi(y)\,dy$$

$$K \to \quad D\varphi(x) = \int \partial_{n_y} \log(|x-y|)\,\varphi(y)\,dy$$

$$K' \to \quad S'\varphi(x) = D^*\varphi(x) = \int_\Gamma \partial_{n_x} \log(|x-y|)\,\varphi(y)\,dy$$

Fredholm Alternative

Theorem 23. (Fredholm Alternative [Kress LIE Thm. 4.14])
*$A: X \to X$ compact. **Then either:***

- *$I - A$ and $I - A^*$ are bijective*

or:

- $\dim N(I - A) = \dim N(I - A^*)$
- $(I - A)(X) = N(I - A^*)^{\perp}$ $\leftarrow$
- $(I - A^*)(X) = N(I - A)^{\perp}$

$(I-A)^* = I - A^*$

$\big((I-A)f, g \big) = (If, g) - (Af, g)$

$N(I - A^*) = \operatorname{span}\{1\}$ $\uparrow$

○ Seen these statements before?
○ Translate to language of integral equation solvability: $f \in \operatorname{span}\{1\}^{\perp}$
○ What about symmetric kernels ($K(x,y) = K(y,x)$)?
○ Where to get uniqueness?

$\Longleftrightarrow$

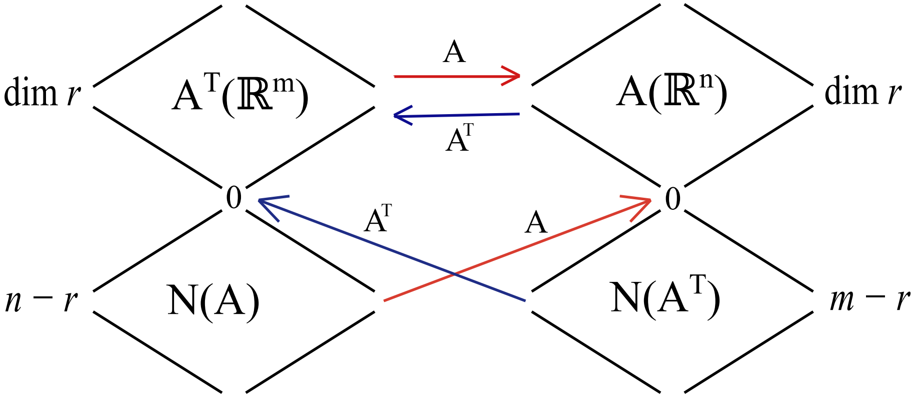
$(f, 1) = 0 \Longleftrightarrow \int f = 0$

$(I - A)\varphi = f$

Fundamental Theorem of Linear Algebra

$A \in \mathbb{R}^{m \times n}$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, bend left, "A"] & \mathbb{R}^m \arrow[l, bend left, "A^T"]
\end{tikzcd}$$



dim r

$A^T(\mathbb{R}^m)$

A

A^T

$A(\mathbb{R}^n)$

dim r

0

A^T

A

0

$n - r$

$N(A)$

$N(A^T)$

$m - r$

$A(x) = N(A^T)^\perp$

6.5 A Tiny Bit of Spectral Theory

Spectral Theory: Terminology

$A\colon X \to X$ bounded, λ is a _ _ _ _ _ value:

Definition 24. (Eigenvalue) *There exists an element* $\phi \in X$, $\phi \neq 0$ *with* $A\phi = \lambda\phi$.

Definition 25. (Regular value) *The "resolvent"* $(\lambda I - A)^{-1}$ *exists and is bounded.*

$\to A\varphi = \lambda\varphi \quad \underbrace{(\lambda I - A)}\varphi = 0$

- ○ Can a value be regular and "eigen" at the same time?
- ○ What's special about ∞-dim here?

Definition 26. (Resolvent set) $\rho(A) := \{\lambda \textit{ is regular}\}$

Definition 27. (Spectrum) $\sigma(A) := \mathbb{C} \setminus \rho(A)$

Spectral Theory of Compact Operators

Theorem 28. *$A\colon X \to X$ compact linear operator, X ∞-dim.*

Then:

- $0 \in \sigma(A)$ *(show!)*
- $\sigma(A) \setminus \{0\}$ *consists only of eigenvalues*
- $\sigma(A) \setminus \{0\}$ *is at most countable*
- $\sigma(A)$ *has no accumulation point except for 0*

- Show first part. → suppose $0 \notin \sigma(A) \Rightarrow (\ \ -A)^{-1} \Rightarrow \underset{\substack{\uparrow \\ \text{compact}}}{A}A^{-1} = I$
- Show second part.
- Rephrase last two: how many eigenvalues with $|\cdot| \geq R$?
- **Recap:** What do compact operators do to high-frequency data?
- Don't confuse $I - A$ with A itself!

7 Singular Integrals and Potential Theory

Recap: Layer potentials

$$(S\sigma)(x) := \int_\Gamma G(x-y)\,\sigma(y)\,ds_y$$

$$(S'\sigma)(x) := PV\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\,\sigma(y)\,ds_y$$

$$(D\sigma)(x) := PV \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y$$

$$(D'\sigma)(x) := f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y$$

Definition 29. (Harmonic function) $\triangle u = 0$

- Where are layer potentials harmonic?

On the double layer again

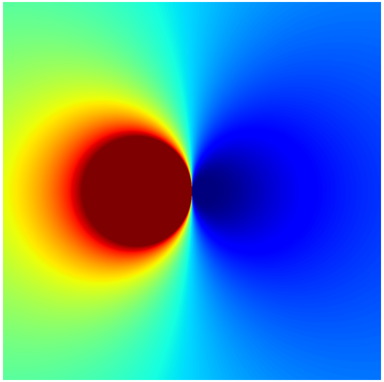
Is the double layer *actually* weakly singular?
Recap:

Definition 30. (Weakly singular kernel)

- *K defined, continuous everywhere except at $x = y$*
- *There exist $C > 0$, $\alpha \in (0, n-1]$ such that*

$$|K(x,y)| \leq C\,|x-y|^{\alpha-n+1} \qquad (x, y \in \partial\Omega,\ x \neq y)$$

$$\frac{\partial}{\partial_x}\log(|0-x|)=\frac{x}{x^2+y^2}$$



- Singularity with approach on $y=0$?
- Singularity with approach on $x=0$?

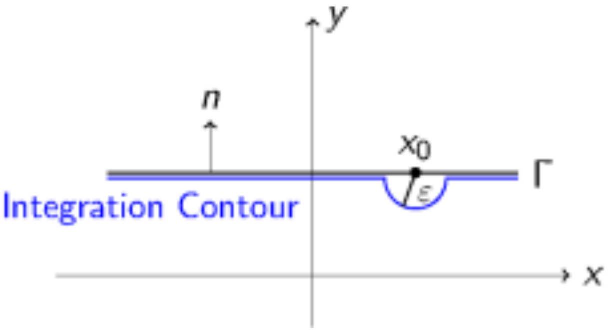
So life is simultaneously worse and better than discussed.
How about 3D? $(-x/|x|^3)$

Would like an analytical tool that requires ‘less’ fanciness.

Cauchy Principal Value

- But I don’t **want** to integrate across a singularity!
- $$\int_{-1}^{1} \frac{1}{x}\, dx?$$

Principal Value in n dimensions



- Again: Symmetry matters!
- What about even worse singularities?

Recap: Layer potentials

$$
\begin{aligned}
(S\sigma)(x) &:= \int_\Gamma G(x-y)\,\sigma(y)\,ds_y \\
(S'\sigma)(x) &:= \mathrm{PV}\ \hat{n}\cdot\nabla_x \int_\Gamma G(x-y)\,\sigma(y)\,ds_y \\
(D\sigma)(x) &:= \mathrm{PV} \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y \\
(D'\sigma)(x) &:= f.p.\ \hat{n}\cdot\nabla_x \int_\Gamma \hat{n}\cdot\nabla_y G(x-y)\,\sigma(y)\,ds_y
\end{aligned}
$$

Important for us: Recover ‘average’ of interior and exterior limit without having to refer to off-surface values.

Green's Theorem

Theorem 31. (Green's Theorem [Kress LIE Thm 6.3])

$$\int_{\Omega} u \triangle v + \nabla u \cdot \nabla v = \int_{\partial\Omega} u \left(\hat{n} \cdot \nabla v\right) ds$$

$$\int_{\Omega} u \triangle v - v \triangle u = \int_{\partial\Omega} u \left(\hat{n} \cdot \nabla v\right) - v \left(\hat{n} \cdot \nabla u\right) ds$$

- If $\triangle v = 0$, then
$$\int_{\partial\Omega} \hat{n} \cdot \nabla v = ?$$
- What if $\triangle v = 0$ and $u = G(|y - x|)$ in Green's second identity?

Green's Formula

Theorem 32. **(Green's Formula** **[Kress LIE Thm 6.5]****)** *If* $\triangle u = 0$, *then*

$$(S\,(\hat{n}\cdot\nabla u) - D\,u)(x) = \begin{cases} u(x) & x \in D \\ \dfrac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

- Suppose I know ‘Cauchy data’ $(u|_{\partial D}, \hat{n}\cdot\nabla u|_{\partial D})$ of u. What can I do?
- What if D is an exterior domain?

Things harmonic functions (don't) do

Theorem 33. (Mean Value Theorem [Kress LIE Thm 6.7]) *If* $\Delta u = 0$,

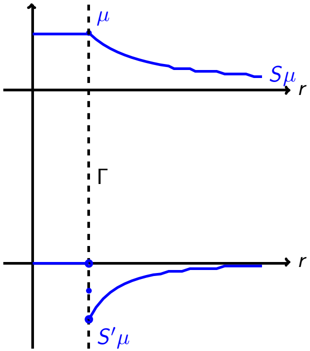
$$u(x) = \fint_{B(x,r)} u(y)\, dy = \fint_{\partial B(x,r)} u(y)\, dy$$

- Define $\fint$?
- Trace back to Green's Formula (say, in 2D):

Theorem 34. (Maximum Principle [Kress LIE 6.9]) *If* $\triangle u = 0$ *on compact set* $\bar{D}$:
u attains its maximum on the boundary.

- Suppose it were to attain its maximum somewhere inside an open set...
- What do our constructed harmonic functions (i.e. layer potentials) do there?

Jump relations



Let $[X] = X_+ - X_-$. (Normal points towards "+"="exterior".)

[Kress LIE Thm. 6.14, 6.17,6.18]

$$[S\sigma] = 0$$

$$\lim_{x \to x_0 \pm} (S'\sigma) = \left(S' \mp \frac{1}{2} I \right)(\sigma)(x_0) \qquad \Rightarrow \qquad [S'\sigma] = -\sigma$$

$$\lim_{x \to x_0 \pm} (D\sigma) = \left(D \pm \frac{1}{2} I \right)(\sigma)(x_0) \qquad \Rightarrow \qquad [D\sigma] = \sigma$$

$$[D'\sigma]=0$$

- Truth in advertising: Assumptions on Γ?

Green's Formula at Infinity (skipped)

$\Omega \subseteq R^n$ bounded, C^1, connected boundary, $\triangle u = 0$, u bounded

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + (S_{\partial B_r}(\hat{n} \cdot \nabla u) - D_{\partial B_r} u)(x) = u(x)$$

for x between $\partial\Omega$ and B_r.

Now $r \to \infty$.

Behavior of individual terms?

Theorem 35. (Green's Formula in the exterior [Kress LIE Thm 6.10])

$$(S_{\partial\Omega}(\hat{n} \cdot \nabla u) - D_{\partial\Omega} u)(x) + \mathrm{PV}\, u_\infty = u(x)$$

for some constant u_∞. Only for $n = 2$,

$$u_\infty = \frac{1}{2\pi r} \int_{|y| = r} u(y)\, ds_y.$$

Theorem 36. (Green's Formula in the exterior [Kress LIE Thm 6.10])

$$(S_{\partial\Omega}(\hat{n}\cdot\nabla u) - D_{\partial\Omega}u)(x) + u_\infty = u(x)$$

- ○ Realize the power of this statement:
- ○ Behavior of the fundamental solution as $r \to \infty$?
- ○ How about its derivatives?

8 Boundary Value Problems

8.1 Laplace

Boundary Value Problems: Overview

	Dirichlet	**Neumann**
Int.	$\lim_{x\to\partial\Omega-} u(x) = g$ ⊕ unique	$\lim_{x\to\partial\Omega-} \hat{n}\cdot\nabla u(x) = g$ ⊙may differ by constant
Ext.	$\lim_{x\to\partial\Omega+} u(x) = g$ $u(x) = \begin{cases} O(1) & 2D \\ o(1) & 3D \end{cases}$ as $\lvert x\rvert\to\infty$ ⊕unique	$\lim_{x\to\partial\Omega+} \hat{n}\cdot\nabla u(x) = g$ $u(x) = o(1)$ as $\lvert x\rvert\to\infty$ ⊕unique

with $g\in C(\partial\Omega)$.

- ○ What does $f(x) = O(1)$ mean? (and $f(x) = o(1)$?)
- ○ Dirichlet uniqueness: why?
- ○ Neumann uniqueness: why?
- ○ Truth in advertising: Missing assumptions on Ω?
- ○ What's a DtN map?

Next mission: Find IE representations for each.

Uniqueness of Integral Equation Solutions

Theorem 37. (Nullspaces [Kress LIE Thm 6.20])

- $N(I/2-D) = N(I/2-S') = \{0\}$
- $N(I/2+D) = \mathrm{span}\{1\}$, $N(I/2+S') = \mathrm{span}\{\psi\}$, *where* $\int \psi \neq 0$.

- ○ Show $N(I/2-D) = \{0\}$.
- ○ Show $N(I/2-S') = \{0\}$.
- ○ Show $N(I+D) = \text{span}\{1\}$.
- ○ What extra conditions on the RHS do we obtain?

→ “Clean” Existence for 3 out of 4.

Patching up Exterior Dirichlet (skipped)

Problem: $N(I/2+S') = \{\psi\}$...but we do not know ψ.
Use a different kernel:

$$\hat{n}\cdot\nabla_y G(x,y) \qquad \to \qquad \hat{n}\cdot\nabla_y G(x,y) + \frac{1}{|x|^{n-2}}$$

Note: Singularity only at origin! (assumed $\in\Omega$)

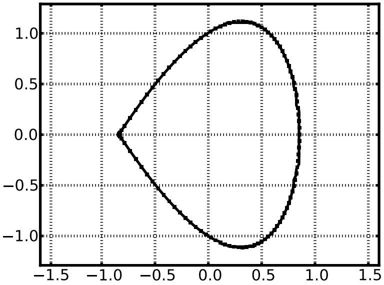
- 2D behavior? 3D behavior?
- Still a solution of the PDE?
- Compact?
- Jump condition? Exterior limit? Deduce $u = 0$ on exterior.
- $|x|^{n-2}u(x) = ?$ on exterior
- Thus $\int \phi = 0$. Contribution of the second term?
- $\phi/2 + D\phi = 0$, i.e. $\phi \in N(I/2+D) = ?$

- Existence/uniqueness?

→ Existence for 4 out of 4.

○ Remaining key shortcoming of IE theory for BVPs?

Domains with Corners



What's the problem? (Hint: Jump condition for constant density)

At corner x_0: (2D)

$$\lim_{x \to x_0 \pm} = \int_{\partial\Omega} \hat{n} \cdot \nabla_y G(x, y)\, \phi(y)\, ds_y \pm \frac{1}{2} \frac{\langle \text{opening angle on } \pm \text{ side} \rangle}{\pi} \phi$$

→ non-continuous behavior of potential on Γ at x_0

What space have we been living in?

Fixes:

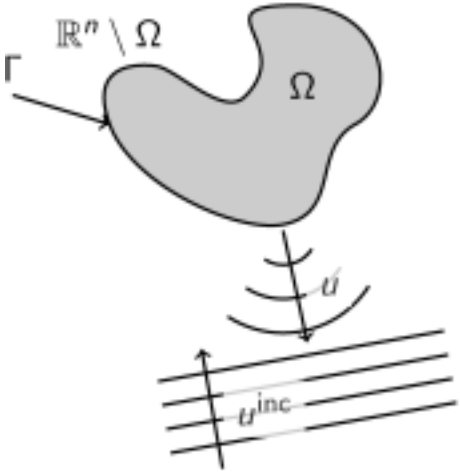
- I + Bounded (Neumann) + Compact (Fredholm)

- Use $\mathbb{L}^2$ theory
  (point behavior “invisible”)

Numerically: Needs consideration, but ultimately easy to fix.

8.2 Helmholtz

The prototypical Helmholtz BVP: A Scattering Problem



Ansatz:

$$u^{\text{tot}} = u + u^{\text{inc}}$$

Solve for scattered field u.

Helmholtz: Some Physics

Physical quantities:

- Velocity potential: $U(x,t) = u(x)\, e^{-i\omega t}$
  (fix phase by e.g. taking real part)
- Velocity: $v = (1/\rho_0)\, \nabla U$
- Pressure: $p = -\partial_t U = i\omega\, u\, e^{-i\omega t}$
  - ○ Equation of state: $p = f(\rho)$

○ What's ρ_0?

○ What happens to a pressure BC as $\omega \to 0$?

Helmholtz: Boundary Conditions

- **Sound-soft**: Pressure remains constant
  - Scatterer “gives”
  - $u = f$ → Dirichlet
- **Sound-hard**: Pressure same on both sides of interface
  - Scatterer “does not give”
  - $\hat{n} \cdot \nabla u = 0$ → Neumann
- **Impedance**: Some pressure translates into motion
  - Scatterer “resists”
  - $\hat{n} \cdot \nabla u + i k \lambda u = 0$ → Robin $(\lambda > 0)$
- **Sommerfeld** radiation condition: allow only outgoing waves

$$r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - i k \right) u(x) \to 0 \qquad (r \to \infty)$$

Many interesting BCs → many IEs! :)

- Transmission between media: What's continuous?

Unchanged from Laplace

Theorem 38. (Green's Formula [Colton/Kress IAEST Thm 2.1]**)** *If* $\triangle u + k^2 u = 0$, *then*

$$(S(\hat{n}\cdot\nabla u) - Du)(x) = \begin{cases} u(x) & x \in D \\ \frac{u(x)}{2} & x \in \partial D \\ 0 & x \notin D \end{cases}$$

$$[Su] = 0$$

$$\lim_{x\to x_0\pm}(S'u) = \left(S' \mp \frac{1}{2}I\right)(u)(x_0) \quad \Rightarrow \quad [S'u] = -u$$

$$\lim_{x\to x_0\pm}(Du) = \left(D \pm \frac{1}{2}I\right)(u)(x_0) \quad \Rightarrow \quad [Du] = u$$

$$= 0$$

- ○ Why is singular behavior (esp. jump conditions) unchanged?
- ○ Why does Green's formula survive?

Remember Green's theorem:

$$\int_{\Omega} u \triangle v - v \triangle u = \int_{\partial\Omega} u\left(\hat{n} \cdot \nabla v\right) - v\left(\hat{n} \cdot \nabla u\right) ds$$

Resonances

$-\triangle$ on a bounded (interior) domain with homogeneous Dirichlet/Neumann BCs has countably many real, positive eigenvalues.

- What does that have to with Helmholtz?
- Why could it cause grief?

Helmholtz: Boundary Value Problems

Find $u \in C(\bar{D})$ with $\triangle u + k^2 = 0$ such that

	Dirichlet	**Neumann**
Int.	$\lim_{x \to \partial D-} u(x) = g$ ⊙unique ($-$resonances)	$\lim_{x \to \partial D-} \hat{n} \cdot \nabla u(x) = g$ ⊙unique ($-$resonances)
Ext.	$\lim_{x \to \partial D+} u(x) = g$ Sommerfeld ⊕unique	$\lim_{x \to \partial D+} \hat{n} \cdot \nabla u(x) = g$ Sommerfeld ⊕unique

with $g \in C(\partial D)$.

- Find layer potential representations for each.

Patching up resonances

Issue: Ext. IE inherits non-uniqueness from ‘adjoint’ int. BVP

Fix: Tweak representation [Brakhage/Werner ‘65, ...]
(also called the ‘CFIE’–‘combined field integral equation’)

$$u = D\phi - i\alpha S\phi$$

(α: tuning knob → 1 is fine, $\sim k$ better for large k)

- ○ How does this help? (skipped)
- ○ Uniqueness for remaining IEs similar. (skipped)

8.3 Calderón identities

- Show that D' is self adjoint.
- Show that $(S\varphi, D'\psi) = ((S' + I/2)\,\varphi, (D - I/2)\,\psi)$.
- $(\varphi, S\,D'\psi)$?

Calderón Identities: Summary

- $SD' = D^2 - I/4$
- $D'S = S'^2 - I/4$

Also valid for Laplace (jump relation same after all!)

- Why do we care?

(to end lec 18)

9 Back from Infinity: Discretization

10 Computing Integrals: Approaches to Quadrature

11 Going General: More PDEs