FFTW: AN ADAPTIVE SOFTWARE ARCHITECTURE FOR THE FFT

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Motivation

• FFT literature has mostly focused on algorithms that minimize the number of floating-point operations.

• On present-day computers, interactions with the processor pipeline and memory hierarchy have a larger impact on performance than number of floating-point operations.

• Propose an adaptive FFT program that tunes the computation automatically for any particular hardware.
Overview

- FFTW's main components: executor, codelets and planner.
- Executor (runtime)
  - Performs the computation of the transform by applying a combination of codelets specified by planner
- Codelets (compile time)
  - Specialized piece of code that computes part of the transform
  - Generated automatically during compile time using FFTW's codelet generator written in Caml Light
- Planner (runtime)
  - Determined during runtime before computation to construct a fast composition of codelets
  - Aims at minimizing actual execution time and not the number of floating point operations
Overview

- User interacts with FFTW only through planner and executor
- Codelet generator is not used after compile time
  - user does not need to know Caml Light or need a Caml Light compiler
- FFTW creates a plan for a transform of a specified size and is reusable as many times as needed

```c
fftw_plan plan;
COMPLEX A[n], B[n];

/* plan the computation */
plan = fftw_create_plan(n);

/* execute the plan */
fftw(plan, A);

/* the plan can be reused for other inputs of size N */
fftw(plan, B);
```
Runtime structure: Executor

$$X_k = \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} nk}$$

\[
X_{N_2 k_1 + k_2} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{N_1 n_2 + n_1} e^{-\frac{2\pi i}{N_1 N_2} \cdot (N_1 n_2 + n_1) \cdot (N_2 k_1 + k_2)}
\]

- Executor implements Cooley-Tukey FFT algorithm
  - Factors the size $N$ of the transform into $N = N_1 N_2$
  - Recursively computes $N_1$ transforms of size $N_2$
  - Multiply the results by 'twiddle factors'
- Computes $N_2$ transforms of size $N_1$
- The algorithm mainly composed of two codelet variations (SIMD supported)
  - **Normal codelets**: Computes DFT of a fixed size and used as base case for recursion
  - **Twiddle codelets**: Like normal codelets, except they multiply their input by the twiddle factors

Runtime structure: Executor

- Codelet with SIMD support
- Need transpose for appropriate data layout
- Example SIMD scheme: $\text{DFT}(A + iB) = \text{DFT}(A) + i \text{DFT}(B)$

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Runtime structure: Executor

- **Input:**
  1. Plan that specifies data structure of factorization and codelets to use
  2. Array to be transformed

- **Example of a possible plan for a transform of length N=128**
  - Computes 4 transforms of size 32 recursively then uses *twiddle codelet* of size 4 to combine results of the subproblems
  - Computes 8 transforms of size 4 solved directly using *normal codelet* and combined using size 8 twiddle codelet
Runtime structure: Executor

- Implemented with explicit recursion instead of loop-based
  - Divide-and-conquer algorithms improve locality
  - Codelet performs significant amount of work - recursion overhead is negligible
  - Easier to code and allows codelet to perform well-defined task independent of the context
Runtime structure: Planner

- Strategy: Construct plan with combinations of codelets and measure execution time of different plans to select the best (ideally, all possible plan)
- Problem: Impractical due to combinatorial explosion of number of plans
- Solution: Use dynamic programming algorithm to reduce search space
  - Assume optimal sub-structure: *If an optimal plan for a size N is known, this plan is still optimal when size N is a subproblem of a larger transform*
  - In theory, the assumption is not true - cache states may differ
  - In practice, simplifying hypothesis yielded good results
Runtime structure: Planner

- Fastest plan is not one that performs the fewest operations
- Total number of flops is not enough to predict execution time
- Optimal plan depends on processor, memory architecture and compiler
- $N = 1024$ is factored into $8 \times 8 \times 16$ on UltraSPARC and into $32 \times 32$ on Alpha

"MFLOPS" defined for a transform of size $N$ as $(5N \log_2 N)/t$
Compile time: Codelet

• Codelet generator is written in Caml Light dialect of ML and is used during compile time
• Input: Size N
• Output: normal or twiddle codelet that performs Fourier transform of size N
• Operates on a subset of abstract syntax tree (AST) of the C language
• Codelet generation is broken down into three phases:
  • Generation: Creates a crude AST, contains *useless code*
  • Optimization/Simplification: Polish and apply local optimization on the crude AST
  • Scheduler: Topological sort of the AST to minimize register spills
  • Translation: Unparse the AST to produce desired C code
Codelet: Generation

- AST generator builds syntax tree recursively
  - Generator needs to decide which algorithm to use at each stage of recursion
    - split-radix - recursive split to N/2-N/4-N/4
    - prime factor - $N = N_1N_2$ where $N_1N_2$ prime numbers
    - Cooley-Tukey - $N = N_1N_2$
    - Rader's algorithm - Computes DFT of prime sizes
  - Minimize a certain cost function which depends on arithmetic complexity and memory traffic
    - Example: $\text{cost} = 4\nu + f$ (experimentally showed good results)
      - $f$ is the number of floating-point operations
      - $\nu$ is the number of stack variable
Codelet: Generation

\[ \sum_{n_1=0}^{N_1-1} \left[ e^{-\frac{2\pi i}{N} n_1 k_2} \right] \left( \sum_{n_2=0}^{N_2-1} x_{N_1 n_2 + n_1} e^{-\frac{2\pi i}{N_2} n_2 k_2} \right) e^{-\frac{2\pi i}{N_1} n_1 k_1} \]

let rec cooley_tukey n1 n2 input sign =
let tmp1 j2 = fftgen n1
    (fun j1 -> input (j1 * n2 + j2)) sign in
let tmp2 i1 j2 =
    exp n (sign * i1 * j2) @* tmp1 j2 i1 in
let tmp3 i1 = fftgen n2 (tmp2 i1) sign in
    (fun i -> tmp3 (i mod n1) (i / n1))

C translation of an AST for a complex DFT of size 2

tmp1 = REAL(input[0]);
tmp5 = REAL(input[0]);
tmp6 = IMAG(input[0]);
tmp2 = IMAG(input[0]);
tmp3 = REAL(input[1]);
tmp7 = REAL(input[1]);
tmp8 = IMAG(input[1]);
tmp4 = IMAG(input[1]);
REAL(output[0]) = ((1 * tmp1) - (0 * tmp2))
    + ((1 * tmp3) - (0 * tmp4));
IMAG(output[0]) = ((1 * tmp2) + (0 * tmp1))
    + ((1 * tmp4) + (0 * tmp3));
REAL(output[1]) = ((1 * tmp5) - (0 * tmp6))
    + ((-1 * tmp7) - (0 * tmp8));
IMAG(output[1]) = ((1 * tmp6) + (0 * tmp5))
    + ((-1 * tmp8) + (0 * tmp7));

Fragment of codelet generator that implements Cooley-Tukey
Codelet: Simplification

- Optimizer consists of a set of rules applied locally to each node in the AST to transform it into one that executes faster.
- Codelet may contain many floating-point constant coefficients pair (i.e. a,-a) from trigonometric identities.
  - Hack: Have a rule to make all constants positive and propagate the minus sign accordingly.
- Floating point constants are typically not part of the program code and are loaded from memory.

```ocaml
let rec stimesM = function
 | (Uminus a, b) -> (* -a * b ==> -(a * b) *)
   stimesM (a, b) >>= suminusM
 | (a, Uminus b) -> (* a * -b ==> -(a * b) *)
   stimesM (a, b) >>= suminusM
 | (Num a, Num b) -> (* multiply two numbers *)
   snumM (Number.mul a b)
 | (Num a, Times (Num b, c)) ->
   snumM (Number.mul a b) >>= fun x ->
   stimesM (x, c)
 | (Num a, b) when Number.is_zero a ->
   snumM Number.zero (* 0 * b ==> 0 *)
 | (Num a, b) when Number.is_one a ->
   returnM b (* 1 * b ==> b *)
 | (Num a, b) when Number.is_mone a ->
   suminusM b (* -1 * b ==> -b *)
 | (a, (Num _ as b')) -> stimesM (b', a)
 | (a, b) -> returnM (Times (a, b))
```
Codelet: Scheduler

- Aim at maximizing register usage (Remember Hong and Kung?)
- However, codelet generator does not address instruction scheduling problem - pipelining
- Heuristic: Recursive partitioning

Illustration of a scheduling problem for FFT on 8 inputs
Compile time: Codelet

- Advantages of codelet generator:
  - Produce correct code automatically
  - Allows hacks such as propagation of the minus sign implemented with minimal code
  - Algorithm and coding style for best performance is not known a priori, generator helps produce and experiment code quickly
Performance results

- Compared FFTW with over 40 other complex FFT implementations on 7 platforms (not all is shown)
- Obtained similar numbers on other machines
  - On IBM RS/6000, comparing with IBM's ESSL library
    - N = 64, FFTW 55% faster
    - N=16384, FFTW 12% slower
    - N = 131072, FFTW 7% faster
Conclusion

- Manually optimizing software is impractical due to complexity of computer architecture
- FFTW provides a method to address such complexity by minimizing execution time instead of arithmetic complexity
  - Planner seeks for best execution plan
  - Codelet generator generates optimized code for transforms