11/29/2006. Final Project: due 12/13 3PM.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \qquad f(u) = \begin{pmatrix} f_1(u_1, \cdots u_m) \\ \\ f_m(u_1 \cdots u_m) \end{pmatrix}$$

Hyperbolic :

$$f'(u) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1}, & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_m}{\partial u_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial u_1} & \frac{\partial f_m}{\partial u_2} & \frac{\partial f_m}{\partial u_m} \end{pmatrix}$$

$\Rightarrow$ real eigenvalues and a complete set of eigenvectors.

right e

e-values $\lambda_1 \ \lambda_2 \cdots \lambda_m$ not distinct. $f'(u) r_i = \lambda_i r_i$

e-vectors $r_1 \ r_2 \ \ r_m$ independent column vectors.

$R = (r_1 \cdots r_m)$

$R^{-1} = \begin{pmatrix} l_1 \\ \vdots \\ l_m \end{pmatrix}$ l_is are row vectors.

left eigenvector $l_i f'(u) = \lambda_i l_i$

(last class $u_t + A u_x = 0$) scalar case.

Converting from 1D $\rightarrow$ 2D.

e.g. Finite volume

we are given $\{\bar{u}_j\}$ — @ each grid pt cell avg is a vector.

$$\frac{d\bar{u}_j}{dt} + \frac{1}{\Delta x} \left[\hat{f}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}) - \hat{f}(u^-_{j-\frac{1}{2}}, u^+_{j-\frac{1}{2}}) \right]$$

We do not know exact solution so we don't know anything about the solution but when you compute numerical solution looks good. So we trust it.

How to do

(i) $\hat{f}$?

(2) $\{\bar{u}_j\} \xrightarrow{\text{reconstruction}} \{u^{\pm}_{j+\frac{1}{2}}\}$

(i) $\hat{f}(u^-, u^+)$ are called (or obtained from) (apx) Riemann Solvers.

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = \begin{cases} u^- & x<0 \\ u^+ & x \geq 0 \end{cases} \end{cases}$$

If you can get the exact solution, $u(x,t) = V(x/t)$ just like scalar case.

then the flux $\hat{f}(u^-, u^+) = f(V(0))$ ← Godunov scheme.

(*) Most of the time this is not realistic.

since sometimes you cant find exact solution ie $V(0)$.

(iia) It doesn't matter for linear case ∵ eigenvalues are const. but in this case $f'(u)$ depends on u ⇒ eigenvalues eigenvectors will be different for diff $j+\frac{1}{2}$ location.

At $x_{j+\frac{1}{2}}$ find a "reference" vector,

- $\tilde{u}_{j+\frac{1}{2}}$ (a crude apx to get the matrices) base on $\bar{u}_j, \bar{u}_{j+1}$.

e.g.) $\tilde{u}_{j+\frac{1}{2}} = \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1})$

b) Roe avg (good because he is alive)

$$f(\bar{u}_{j+1}) - f(\bar{u}_j) = f'(\tilde{u}_{j+\frac{1}{2}})(\bar{u}_{j+1} - \bar{u}_j)$$

if you can find .. might not exist ⇒ documented iib) R, R^{-1} $\quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$ based on $f'(\tilde{u}_{j+\frac{1}{2}})$

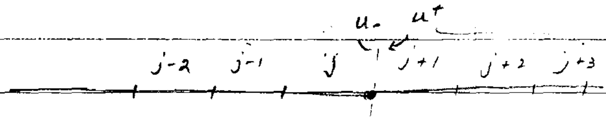
THERE ARE FORMULAE

iic) Project all the cell averages which might be used to evaluate $u^{\pm}_{j+\frac{1}{2}}$ to the local characteristic fields.

(*) $R^{-1}\bar{u}_i = \bar{v}_i$ $\quad \Big| \; i = j-2 \; \cdots \; j+3$ ← These indices are specific for 3rd order ENO or 5th order WENO

For each $j+\frac{1}{2}$.

Very costly to do for all.



So frozen @ $j+\frac{1}{2}$ then do for i.

iid) The usual scalar reconstruction on each component of $\{\bar{v}_i\}$

(*) SAME AS THE SCALAR SUBROUTINE

$v^{\pm}_{j+\frac{1}{2}}$

iie) Recover the reconstruction

$$u^{\pm}_{j+\frac{1}{2}} = R\, v^{\pm}_{j+\frac{1}{2}}$$

This procedure is meaningful if you have a non linear reconstruction.

(ENO, WENO)

eg Linear reconstruction using 2 cell avgs. $v^-_{j+\frac{1}{2}} = \frac{1}{2}(\bar{v}_j + \bar{v}_{j+1})$

new method

$\longrightarrow \; u^-_{j+\frac{1}{2}} = \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1})$

$\bar{v}_i = R^{-1}\bar{u}_i, \; i = j, j+1$

$v^-_{j+\frac{1}{2}} = \frac{1}{2}(\bar{v}_j + \bar{v}_{j+1})$, $u^-_{j+\frac{1}{2}} = R\left(\frac{1}{2}(\bar{v}_j + \bar{v}_{j+1})\right)$

$= \frac{1}{2}(R\bar{v}_j + R\bar{v}_{j+1})$

$= \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1})$

2D system

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0 \\ u(x,y,0) = u^0(x,y) \end{cases}$$

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$ $\quad f(u) = \begin{pmatrix} f_1(u_1 \cdots u_m) \\ \vdots \\ f_m(u_1 \cdots u_m) \end{pmatrix}$

$g(u) = \begin{pmatrix} \\ \\ \end{pmatrix}$

Hyperbolicity

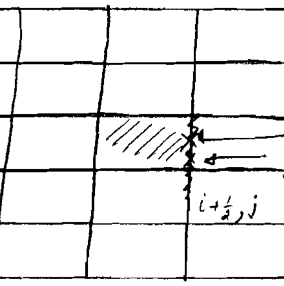
$\xi_1 f'(u) + \xi_2 g'(u)$ $\quad$ ξ_1, ξ_2 are real #'s

$\uparrow$ JACOBIAN $\quad$ $\xi_1^2 + \xi_2^2 \neq 0$

has real eigenvalues and a complete set of eigenvectors.

FINITE VOLUME SCHEMES.

- Rectangular meshes



If u choose 2 gaussian pts will give you 4th order accuracy

$$\frac{d\tilde{u}_{ij}}{dt} + \frac{1}{\Delta x}\left[\hat{f}_{i+\frac{1}{2},j} - \hat{f}_{i-\frac{1}{2},j}\right] + \frac{1}{\Delta y}\left[\hat{g}_{i,j+\frac{1}{2}} - \hat{g}_{i,j-\frac{1}{2}}\right] = 0$$

So say 2 Gauss points, $\quad \alpha = \frac{1}{\sqrt{3}}$ (?)

$$\hat{f}_{i+\frac{1}{2},j} = \frac{1}{2}\left[\hat{f}(u^-_{i+\frac{1}{2},j-\alpha}, u^+_{i+\frac{1}{2},j-\alpha}) + \hat{f}(u^-_{i+\frac{1}{2},j+\alpha}, u^+_{i+\frac{1}{2},j+\alpha})\right]$$

Questions: $\cdot$ $\hat{f}$ (once you have u^-, u^+).

$\cdot$ reconstruction. (How to get the pt values of u)

Ans $\cdot$ The choice of $\hat{f}$ is the same as the 1-D case

Pretend 1D conservation law,

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0 \end{cases} \end{cases}$$

exact or approximate Riemann solver

$v(x/t) \rightarrow \hat{f}(u^-, u^+) = f(v(0))$

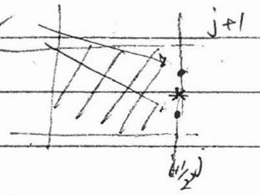
- Reconstruction MESSY!

given $\{\tilde{\bar{u}}_{ij}\} \longrightarrow \{u^{\pm}_{i+\frac{1}{2}, j+\alpha}\}$

reference pt for characteristic

- $u^{*}_{i+\frac{1}{2}, j}$

We are going to use the same one for



so is a 'crude' avg of $\tilde{\bar{u}}_{j}$ & $\tilde{\bar{u}}_{i+1,j}$

eg → either simple avg

$$u^{*}_{i+\frac{1}{2}, j} = \frac{1}{2}\left(\tilde{\bar{u}}_{ij} + \tilde{\bar{u}}_{i+1,j}\right)$$

→ or Roe avg

$$f(\tilde{\bar{u}}_{i+1,j}) - f(\tilde{\bar{u}}_{ij}) = f'(u^{*}_{i+\frac{1}{2},j})(\tilde{\bar{u}}_{i+1,j} - \tilde{\bar{u}}_{i,j})$$

- R, R^{-1} $\quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$ associated with $f'(u^{*}_{i+\frac{1}{2}, j})$

- Project all the cell averages that might be needed in the reconstruction

$$R^{-1}\,\tilde{\bar{u}}_{kl} = \tilde{\bar{V}}_{k,l} \quad \left(\begin{array}{l} k = i-2, \cdots i+3. \text{ indices depend on the} \\ l = j-3, \cdots j+3. \text{ reconstruction, ENO, TVD} \end{array}\right)$$

- Perform scalar reconstruction procedure on each component of $\{\tilde{\bar{V}}_{k,l}\}$ to get $\{v^{\pm}_{i+\frac{1}{2}, j\pm\alpha}\}$ ← do it at the same time, saves computation.

- $u^{\pm}_{i+\frac{1}{2}, j\pm\alpha} = R\, V^{\pm}_{i+\frac{1}{2}, j\pm\alpha}$

- Obtain $\hat{f}_{i+\frac{1}{2}, j}$ (by feeding into apx Rieman solvers)

- Similarly, we get $\hat{g}_{i, j+\frac{1}{2}}$

(*) Majority of the work is 1-D, only 2-D part is reconstruction part.

So you are assuing the flow of flux is 1-D