PDE 1 – Second half

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1 1-D Wave Equation

$$u_{tt} = c^2 u_{xx} = 0 \tag{1.1}$$

for $x \in \mathbb{R}$ an d $t > 0$ with $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. D'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(y)\,dy\right].$$

Geometric identity:

$$u(A) + u(C) = u(B) + u(D). \tag{1.2}$$

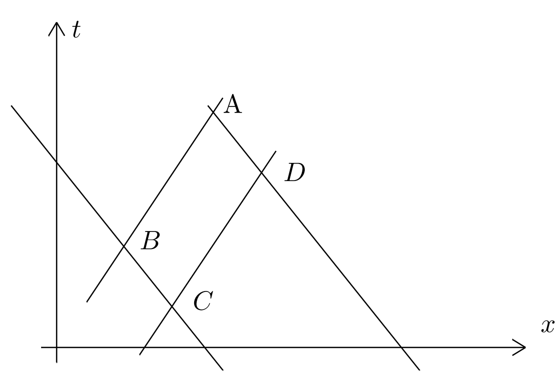

Figure 1.1. Sketch for the geometric identity.

We have: C^2 solution of (1.1) $\Leftrightarrow$ (1.2) for every characteristic parallogram.

1.1 Boundary conditions

Good and bad boundary conditions:

$$0 = u_t + c\,u_x,$$

supposing $c > 0$.

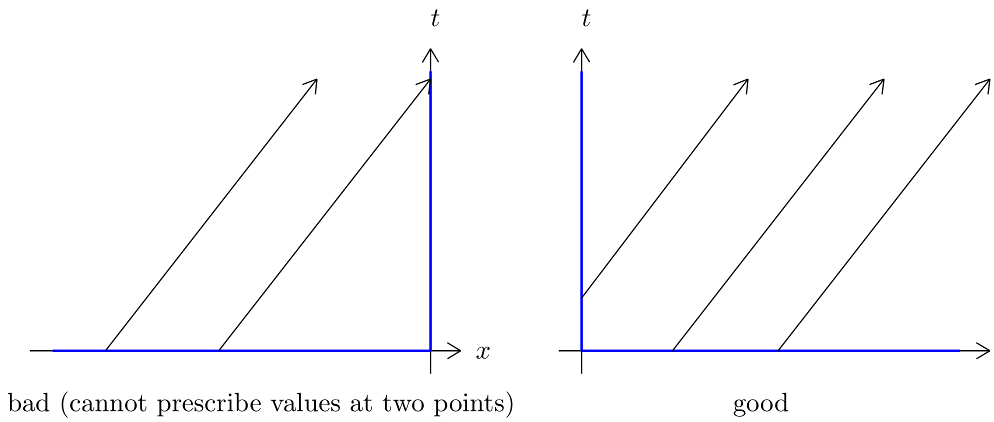

Figure 1.2. Good and bad boundary conditions for the transport equation.

Example:

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in (0, \infty), t > 0$$

$u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$ for $x \in \mathbb{R}$. $u(0, t) = 0$ for $t \geqslant 0$ with the assumption that $f(0) = 0$.

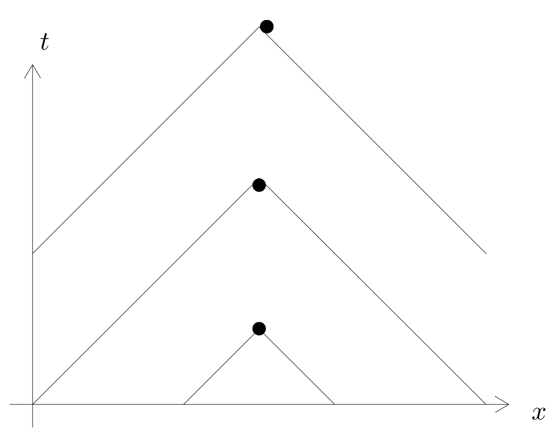

Figure 1.3. Domain of dependence.

The dependency on ICs outside of the domain is solved by the *method of reflection*. Extend u to all of $\mathbb{R}$, say $\tilde{u}$.

$$\tilde{u}(x, t) = \frac{1}{2}\left[\tilde{f}(x + c\,t) + \tilde{f}(x - c\,t) + \frac{1}{c}\int_{x-ct}^{x+ct} \tilde{g}(y)\,dy\right].$$

$$\tilde{u}(0,t) = \frac{1}{2}\left[\tilde{f}(ct) + \tilde{f}(ct) + \frac{1}{c}\int_{ct}^{ct} \tilde{g}(y)\,dy\right].$$

Choose odd extension:

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & x \geqslant 0, \\ -u(-x,t) & x < 0. \end{cases}$$

Similarly for $\tilde{f}$, $\tilde{g}$. Then $\tilde{u}(0,t) = 0 = u(0,t)$. $u(x,t) = \tilde{u}(x,t)$ for $x > 0$.

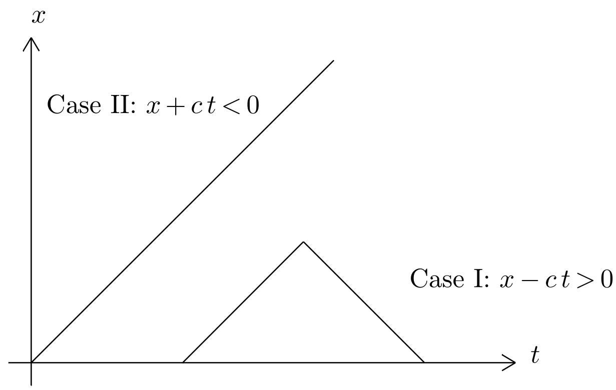

Figure 1.4. Different cases arising for the determination of the domain of dependence.

Case 1: D'Alembert as before.
Case 2:

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + \underbrace{f(ct-x)}_{\text{odd ext.}} + \frac{1}{c}\int_{ct-x}^{x+ct} g(y)\,dy\right].$$

If $g \equiv 0$, this corresponds to reflection as follows:

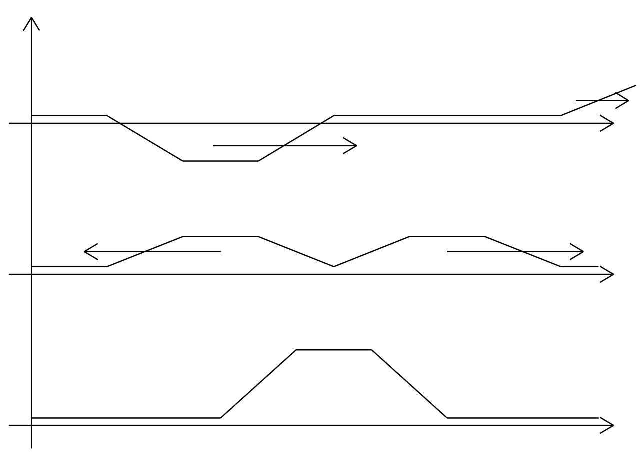

Figure 1.5. Series of snapshots of solutions with $g = 0$.

Initial boundary value problem:

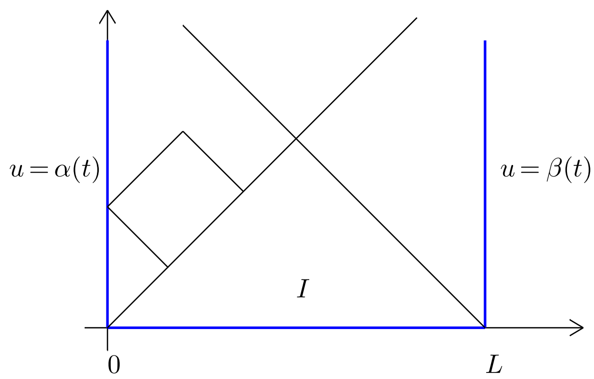

Figure 1.6. Initial boundary value problem. We can satisfy the parallelogram identity using geometry.

For arbitrary α,β the equation need not have a continuous solution:

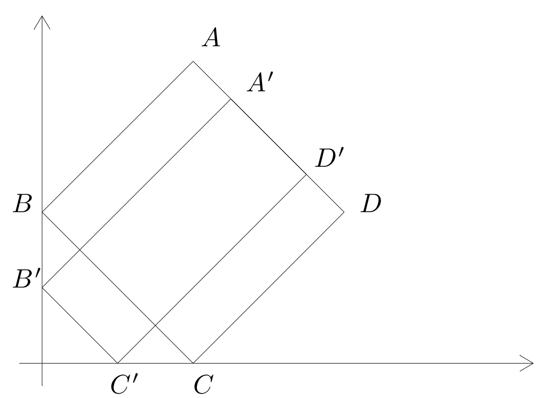

Figure 1.7. Discontinuous solutions in corners.

Assume $u\in C((0,L]\times(0,\infty))$.

$$\begin{aligned} u(B) &= \alpha(B),\\ u(C) &= f(C). \end{aligned}$$

$u(A)+u(C)=u(B)+u(D)$. $A\to D\Rightarrow u(A)\to u(D)$, $u(C)=u(B)\Rightarrow \lim_{t\to 0}\alpha(t)=\lim_{x\to 0} f(x)$. Similarly, if we want $u\in C^1$, this requires $\alpha'(0)=g(0)$, etc.

1.2 Method of Spherical Means

$$\partial_t^2 u - c^2\Delta u = 0$$

for all $x\in\mathbb{R}^n$ and $t>0$ with

$$\begin{aligned} u(x,0) &= f(x),\\ u_t(x,0) &= g(x). \end{aligned}$$

If $h{:}\,\mathbb{R}^n\to\mathbb{R}$, let

$$\begin{aligned} M_h(x,r) &= \frac{1}{\omega_n r^{n-1}}\int_{S(x,r)} h(y)\,dS_y\\ &= \frac{1}{\omega_n}\int_{|\omega|=1} h(x+r\omega)\,dS_\omega. \end{aligned}$$

Assume that h is continuous. Then

1. $\lim_{r\to 0} M_h(x,r) = h(x)$ for every $x \in \mathbb{R}^n$.
2. $M_h(x,r)$ is a continuous and even function.

If $h \in C^2(\mathbb{R}^n)$, then

$$\Delta_x M_h(x,r) = \frac{\partial^2}{\partial r^2} M_h + \frac{n-1}{r}\frac{\partial M_h}{\partial_r}.$$

If you view M_h as a function $M_h\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is spherically symmetric, then the above equation states that the Laplacian in the first n variables equals the Laplacian in the second n. Spherical means of

$$\partial_t^2 u - c^2 \Delta_x u = 0.$$

Then

$$\partial_t^2 M_u - c^2 \Delta_x M_u = 0$$

and

$$\partial_t^2 M_u - \left[\frac{\partial^2}{\partial_r^2 M_u} + \frac{n-1}{r}\frac{\partial M_u}{\partial r}\right] = 0.$$

1.3 Wave equation in $\mathbb{R}^n$

$$\square u := u_{tt} - c^2 \Delta u = 0 \qquad (*)$$

for $x \in \mathbb{R}^n \times (0,\infty)$ with $u = f$ and $u_t = g$ for $x \in \mathbb{R}^n$ and $t = 0$. Now do Fourier analysis: If h $\in L^1(\mathbb{R}^n)$, consider

$$\hat{h}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} h(x) dx.$$

If we take the FT of $(*)$, we get

$$\hat{u}_{tt} + c^2 |\xi|^2 \hat{u} = 0$$

for $\xi \in \mathbb{R}^n$ and $t > 0$, $\hat{u}(\xi,0) = \hat{f}$, $\hat{u}(\xi,0) = \hat{g}$. $\hat{u}(\xi,t) = A\cos(c|\xi|t) + B\sin(c|\xi|t)$. Use ICs to find

$$\hat{u}(\xi,t) \;=\; \hat{f}(\xi)\cos(c|\xi|t) + \hat{g}(\xi)\frac{\sin(c|\xi|t)}{c|\xi|}.$$

Analogous caclulation for heat equation:

$$u_t - u_{xx} = 0 \Rightarrow \hat{u}_t + |\xi|^2 \hat{u} = 0, \quad \hat{u}(\xi,0) = \hat{f}$$

yields $\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{f}(\xi)$. Then observe that multiplication becomes convolution.

Observe that

$$\cos(c|\xi|t) = \partial_t\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right).$$

If we could find a $k(x,t)$ such that

$$\frac{\sin(c|\xi|t)}{c|\xi|} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} k(x,t) dx,$$

this would lead to a solution formula

$$u(x,t) = \int_{\mathbb{R}^n} k(x-y,t) g(y) dy + \partial_t \int_{\mathbb{R}^n} k(x-y,t) f(y) dy.$$

Suppose $n = 1$, we know that our solution formula must coincide with D'Alembert's formula

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(y)\, dy\right].$$

Here

$$\begin{aligned} k(x,t) &= \frac{1}{2c}\mathbf{1}_{\{|x|\leqslant ct\}}, \\ \partial_t k(x,t) &= \frac{1}{2}\left[\delta_{\{x=ct\}} + \delta_{\{x=-ct\}}\right]. \end{aligned}$$

Solution formula for $n=3$:

Theorem 1.1. *$u\in C^\infty(\mathbb{R}^3\times\mathbb{R})$ is a solution to the wave equation with C^∞ initial data f, g if and only if*

$$u(x,t)=\fint_{S(x,ct)}[t\,g(y)+f(y)+Df(y)(y-x)]dS_y.$$

Here,

$$k(x,t)=\frac{1}{4\pi c^2t}\cdot dS_y\underset{|x|=ct}{|}=t\cdot\text{uniform measure on }\{|x|=c\,t\}.$$

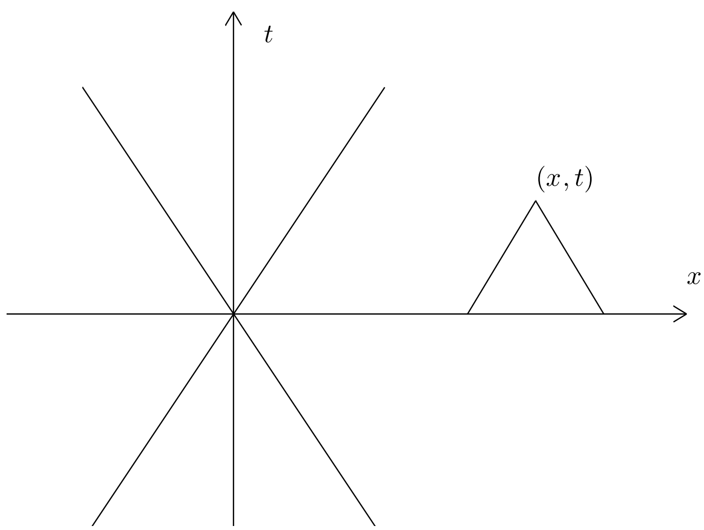

Figure 1.8.

1.4 Method of spherical means

Definition 1.2. *Suppose $h\colon\mathbb{R}^n\to\mathbb{R}$ is continuous. Define $M_h\colon\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ by*

$$M_h(x,r)=\fint_{S(x,r)}h(y)dS_y=\frac{1}{\omega_n}\int_{|\omega|=1}h(x+r\omega)\cdot d\omega.$$

Notice that

$$\lim_{r\to 0}M_h(x,r)=h(x)$$

if h is continuous.

Darboux's equation: Suppose $h\in C^2(\mathbb{R}^n)$. Then

$$\Delta_xM_h(x)=\frac{\partial^2}{\partial r}M_h+\frac{n-1}{r}\cdot\frac{\partial M_h}{r}.$$

Proof. Similar to the mean value property for Laplace's equation.

$$\begin{aligned}\int_0^r\Delta_xM_h(x,\rho)\rho^{n-1}\cdot d\rho &= \int_0^r\Delta_x\frac{1}{\omega_n}\int_{|\omega|=1}h(x+\rho\omega)\cdot d\omega\,\rho^{n-1}d\rho\\ &= \int_{B(0,r)}\Delta_xh(x+y)\cdot dy=\frac{1}{\omega_n}\int_{S(0,r)}\frac{\partial h}{\partial n_y}(x+y)dy\\ (y=r\omega,dy=r^{n-1}d\omega)\quad &= \frac{1}{\omega_n}\int_{S(0,r)}Dh(x+y)\cdot n_y\,dy\\ &= \frac{r^{n-1}}{\omega_n}\int_{|\omega|=1}\frac{d}{dr}(h(x+r\omega)\cdot d\omega=r^{n-1}\frac{\partial M_h}{\partial r}.\end{aligned}$$

Then

$$\int_0^r\Delta_xM_h(x,\rho)\rho^{n-1}\cdot d\rho=r^{n-1}\frac{d}{dr}M_h.$$

Differentiate

$$\begin{aligned}\Delta_x M_h r^{n-1} &= \frac{d}{dr}\left[r^{n-1}\cdot\frac{dM_h}{dr}\right]\\ &= r^{n-1}\cdot\frac{d^2}{dr^2}+(n-1)r^{n-2}\frac{dM_h}{dr}.\end{aligned}$$

Altogether

$$\Delta_x M_h = \frac{\partial^2 M_h}{\partial r^2}+\frac{(n-1)}{r}\frac{\partial M_h}{\partial r}. \qquad \square$$

Look at spherical means of $(*)$:

$$u_{tt}-c^2\Delta u=0$$

Assume $u\in C^2(\mathbb{R}^n\times(0,\infty))$. Take spherical means:

$$M_{u_{tt}}=(M_u)_{tt},$$

which means

$$\begin{aligned}\partial_t^2 ⨍_{S(x,r)} u(y,t)dS_y &= ⨍_{S(x,r)}\partial_t^2 u(y,t)dy,\\ (M_u)_{tt} &= M_{u_{tt}}.\end{aligned}$$

And

$$M_h(\Delta_x u)\overset{\text{Darboux}}{=}\frac{\partial^2 M_h}{\partial r^2}+(n-1)\frac{\partial M_h}{\partial r}.$$

Therefore, we have

$$(M_u)_{tt}=c^2\left[\frac{\partial^2 M_h}{\partial r^2}+(n-1)\frac{\partial M_h}{\partial r}\right].$$

If $n=1$, we can solve by D'Alembert. For $n=3$:

$$\frac{\partial^2}{\partial r^2}(r\,M_h) = \frac{\partial}{\partial r}\left(r\frac{\partial M_h}{\partial r}+M_h\right)=r\frac{\partial^2 M_h}{\partial r^2}+2\cdot\frac{\partial M_h}{\partial r}.$$

So if $n=3$, we have

$$(r\,M_u)_{tt}=c^2\frac{\partial^2}{\partial r^2}(r\,M_h)$$

This is a 1D wave equation (in r!). Solve for $r\,M_h$ by D'Alembert.

$$M_h(x,r,t) = \frac{1}{2r}\left[(r+ct)M_f(x,r+ct)+\underbrace{(r-ct)}_{\text{a)}}M_f(x,r-ct)\right]+\underbrace{\frac{1}{2c\,r}\int_{r-ct}^{r+ct} r' M_g(x,r')dr'}_{\text{b)}}$$

Pass to limit $r\to 0$ in b)

$$\frac{1}{2c\,r}\int_{r-ct}^{r+ct} r'M_g(x,r')dr' = \frac{1}{2c\,r}\int_{ct-r}^{ct+r} r'M_g(x,r')dr'$$

M_g is even, $r\,M_g$ is odd. So

$$\lim_{r\to 0}\text{b)}=\frac{1}{c}\cdot c\,t\,M_g(x,ct)=t\,M_g(x,ct).$$

$$t\,M_g(x,ct)=t⨍_{|x-y|=ct} g(y)dS_y.$$

Similarly, a): (M_f even in r)

$$\begin{aligned} &= \frac{1}{2}[M_f(x,r+ct)+M_f(x,ct-r)]+\frac{1}{2\,r}c\,t\,[M_f(x,ct+r)-M_f(x,ct-r)]\\ \lim_{r\to 0} * &= M_f(x,ct)+c\,t\,\partial_2 M_f(x,ct)=\partial_t(t\,M_f(x,ct)).\end{aligned}$$

For any $\varphi\in C^\infty(\mathbb{R}^3)$ define

$$(K_t*\varphi)(x):=t\int_{|x-y|=ct}\varphi(y)dS_y.$$

Then if $f,g\in C^\infty$, our solution to $\square u=0$ is

$$\boxed{u(x,t)=(K_t*g)(x)+\partial_t(K_t*f)(x).}$$

Aside: Check that

$$\int_{|y|=ct} e^{-i\xi\cdot y} dS_y = \frac{\sin(ct|\xi|)}{c|\xi|}.$$

Remark 1.3. Huygens' principle:

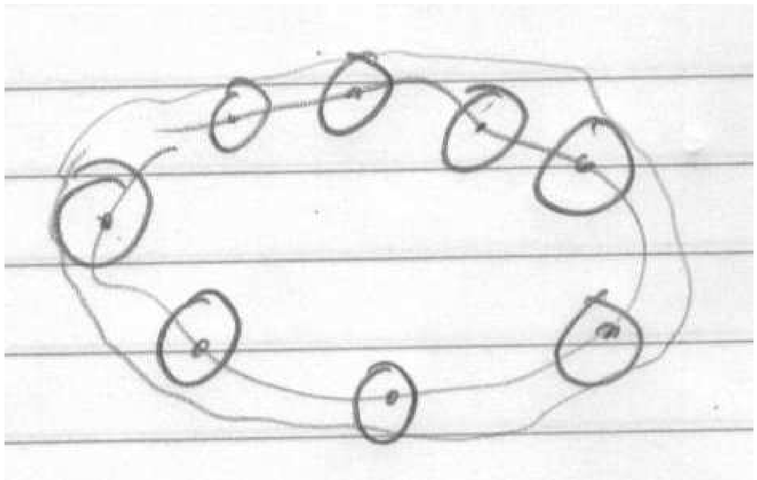

Figure 1.9. Huygens' principle.

We consider data f, g with compact support. Let

$$\Sigma(t) = \operatorname{supp}(u(x,t)) \subset \mathbb{R}^3,$$

where obviously

$$\Sigma(0) = \operatorname{supp}(f) \cup \operatorname{supp}(g).$$

Then Huygens' principle is stated as

$$\Sigma(t) \subset \{x: \operatorname{dist}(x, \Sigma(0)) = ct\}.$$

Example 1.4. Consider radial data g and $f \equiv 0$.

$$u(x,t) = t \fint_{|x-y|=ct} g(y) dS_y.$$

$$u(x,t) \neq 0 \Leftrightarrow S(x,ct) \cap B(0,\rho) \neq \emptyset.$$

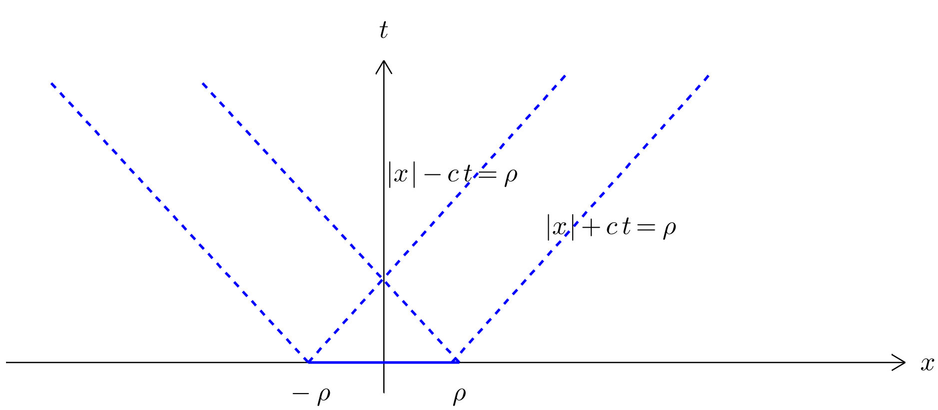

Figure 1.10. How radial data g spreads in time.

Focusing: Assume $g=0$, f radial.

$$\begin{aligned}
u(x,t) &= \partial_t(t\,M_f(x,ct)) = M_f(x,ct) + t\,\partial_t M_f(x,ct)\\
\partial_t M_f(x,ct) &= \partial_t\left(\fint_{|x-y|=ct} f(y)\,dS_y\right)\\
&= \partial_t\left(\fint_{|\omega|=1} f(x+ct\,\omega)\,d\omega\right)\\
&= \fint_{|\omega|=1} Df(x+ct\omega)\cdot(c\,\omega)\,d\omega\\
&= c\fint_{|\omega|=1} \frac{\partial f}{\partial n_\omega}(x+ct\omega)\,d\omega.
\end{aligned}$$

a) b)

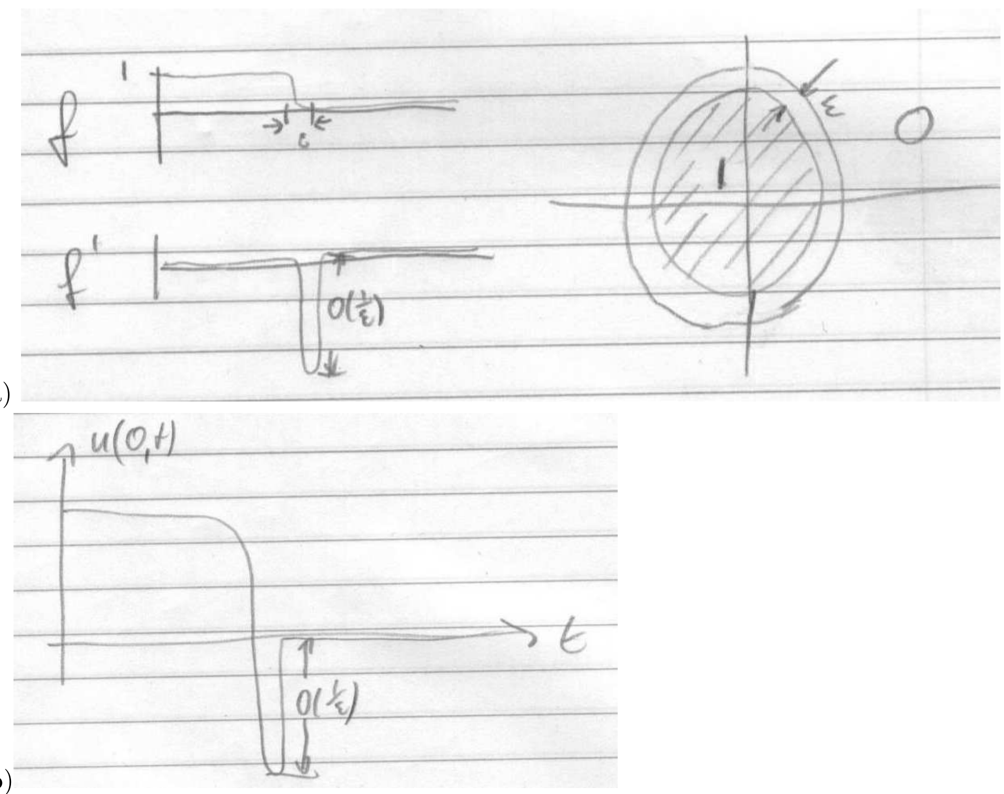

Figure 1.11. a) Spread of data with radial f. b) The sharp dropoff in $u(0,t)$.

$$u(x,t) = \fint_{|x-y|=ct} f(y)\,dS_y + ct\fint_{|x-y|=ct}\frac{\partial f}{\partial n_y}\,dS_y.$$

Thus

$$\|u(x,t)\|_\infty \not\leqslant C\|u(x,0)\|_\infty.$$

More precisely, there exists a sequence $u_0^\varepsilon \in C^\infty(\mathbb{R}^n)$ and t_ε such that

$$\lim_{\varepsilon\downarrow 0}\frac{\sup_x |u^\varepsilon(x,t_\varepsilon)|}{\sup_x |u_0^\varepsilon(x)|} = +\infty.$$

Contrast with solution in $n=1$:

$$\|S(t)u_0\|_{L^p} \leqslant \|u_0\|_{L^p}, \quad 1\leqslant p\leqslant\infty,$$

where $S(t)$ is the shift operator. "=" solution to the wave equation.

Littman's Theorem $S_3(t) =$ solution operator for wave equation in $\mathbb{R}^3$.

$$\sup_{f \in L^p(\mathbb{R}^3)} \frac{\|S_3(t) u_0\|_{L^p}}{\|u_0\|_{L^p}} = +\infty.$$

1.5 Hadamard's Method of Descent

Trick: Treat as 3-dimensional wave equation.

Notation: $x \in \mathbb{R}^2$, $\tilde{x} = (x, x_3) \in \mathbb{R}^3$. If $h: \mathbb{R}^2 \to \mathbb{R}$, define $\tilde{h}: \mathbb{R}^3 \to \mathbb{R}$ by $\tilde{h}(\tilde{x}) = \tilde{h}((x, x_3)) = h(x)$. Suppose u solves $\partial_t^2 u - c^2 \Delta_x u = 0$ for $x \in \mathbb{R}^2$ and $t > 0$ with $u(x, 0) = f(x)$ and $u_t(x, t) = g(x)$. Then

$$\begin{aligned} \partial_t^2 \tilde{u} - c^2 \Delta_{\tilde{x}} &= 0 \\ \tilde{u}(\tilde{x}, 0) &= \tilde{f}(x) \\ \tilde{u}_t(\tilde{x}, 0) &= \tilde{g}(x) \end{aligned}$$

for $\tilde{x} \in \mathbb{R}^3$, $t > 0$.

$$\tilde{u}(\tilde{x}, \tilde{t}) = \partial_t(\tilde{K}_t * \tilde{f}) + \tilde{K}_t * \tilde{g},$$

where

$$=$$

$$\begin{aligned} \tilde{K}_t * \tilde{h} &= t ⨍_{|\tilde{x} - \tilde{y}| = ct} \tilde{h}(y) dS_y \\ &= t ⨍_{|\tilde{\omega}| = 1} \tilde{h}(x + c t \tilde{\omega}) d\tilde{\omega}. \end{aligned}$$

with $\tilde{\omega} \in \mathbb{R}^3 = (\omega, \omega_3)$ for $\omega \in \mathbb{R}^2$. Then

$$\tilde{h}(\tilde{x} + c t \tilde{\omega}) = h(x + c t \omega).$$

$$⨍_{|\tilde{\omega}| = 1} h(x + c t \omega \, d\tilde{\omega}.$$

$\tilde{\omega} = (\omega, \omega_3)$. On $|\tilde{\omega}| = 1$, we have

$$\omega_3 = \pm \sqrt{1 - |\omega|^2} = \pm \sqrt{1 - (\omega_1^2 + \omega_2^2)}.$$

Then

$$\frac{\partial \omega_3}{\partial \omega_i} = \frac{-\omega_i}{\sqrt{1 - |\omega|^2}}$$

for $i = 1, 2$. Thus the Jacobian is

$$\sqrt{1 + \left(\frac{\partial \omega_3}{\partial \omega_1}\right)^2 + \left(\frac{\partial \omega_3}{\partial \omega_2}\right)^2} = \frac{1}{\sqrt{1 - |\omega|^2}}.$$

Thus

$$t ⨍_{|\tilde{\omega}| = 1} h(x + c t \omega) d\tilde{\omega} = \frac{2t}{4\pi} \int_{|\omega| \leqslant 1} \frac{h(x + c t \omega)}{\sqrt{1 - |\omega|^2}} d\omega_1 d\omega_2.$$

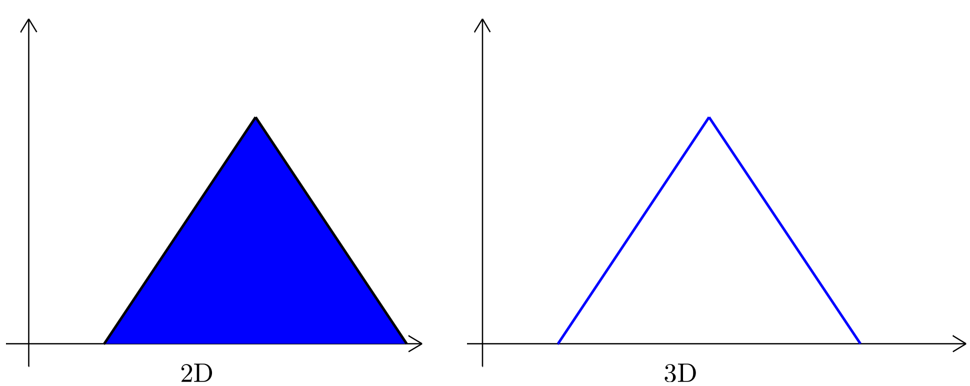

Figure 1.12. Domains of dependence, conceptually, for 2D and 3D.

1.6 Hadamard's Solution for all odd $n \geqslant 3$

[cf. Evans, 2.4] $n = 2k+1$, $k \geqslant 1$. $k = (n-1)/2$, $c = 1$. The general formula is

$$u(x,t) = \partial_t(K_t * f) + K_t * g$$

where for any $h \in C_c^\infty$ we have

$$(K_t * h)(x) = \frac{\omega_n}{\pi^k 2^{k+1}} \left(\frac{1}{t} \cdot \frac{\partial}{\partial t}\right)^{(n-3)/2} \left[t^{n-2} ⨍_{|x-y|=t} h(y) dS_y\right].$$

Check: If $n = 3$, $\omega_n = 4\pi$, so we get our usual formula.

Now, Consider $g \equiv 0$ in $u_{tt} - \Delta u = 0$, $x \in \mathbb{R}^{2k+1}$, $t > 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$. Extend u to $t < 0$ by $u(x,-t) = u(x,t)$ (which is OK because $\partial_t u = 0$ at $t = 0$)

Consider for $t > 0$

$$\begin{aligned} v(x,t) &:= \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-s^2/4t} u(x,s) ds \\ &= \int_{\mathbb{R}} k(s,t) u(x,s)\, ds \quad (*) \end{aligned}$$

Find solution for the heat equation in 1D. Use that $\partial_t k = \partial_s^2 k$.

$$\begin{aligned} \partial_t v &= \int_{\mathbb{R}} \partial_t k\, u(x,s) ds \\ &= \int_{\mathbb{R}} k(s,t) \partial_s^2 u(x,s) ds \\ &= \int_{\mathbb{R}} k(s,t) \Delta_x u(x,s) ds = \Delta_x \int_{\mathbb{R}} k(s,t) u(x,s) ds. \end{aligned}$$

$\partial_t v = \Delta_x v$, $x \in \mathbb{R}^n$, $t > 0$. Also, as $t \to 0$, $v(x,t) \to f(x)$. Therefore,

$$\begin{aligned} v(x,t) &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2/4t} f(x-y) dy \\ &= \frac{1}{(4\pi t)^{n/2}} \int_0^\infty e^{-r^2/4t} r^{n-1} ⨍ f(x - r\omega) \cdot d\omega\, dr \\ &= \frac{\omega_n}{(4\pi t)^{n/2}} \int_0^\infty e^{-r^2/4t} r^{n-1} M_f(x,r) dr \quad (\#) \end{aligned}$$

Change variables using $\lambda = 1/4t$ and equate $(*)$ and $(\#)$:

$$\int_0^\infty e^{-\lambda r^2} u(x,r) dr = \frac{\omega_n}{2} \cdot \frac{1}{\pi^k} \int_0^\infty e^{-\lambda r^2} \lambda^k r^{n-1} M_f(x,r) dr.$$

Then, use the Laplace transform for $h \in L^1(\mathbb{R}_+)$:

$$h^{\#}(\lambda) = \int_0^\infty e^{-\lambda\varphi} h(\varphi) d\varphi.$$

Basic fact: $h^{\#}$ is invertible. Observe that

$$\frac{d}{dr}(e^{-\lambda r^2}) = -\lambda e^{-\lambda r^2}.$$

In particular,

$$\left(-\frac{1}{2r} \cdot \frac{d}{dr}\right)^k e^{-\lambda r^2} = \lambda^k e^{-\lambda r^2}.$$

Therefore

$$\begin{aligned} \int_0^\infty \lambda^k e^{-\lambda r^2} r^{n-1} M_f(x,r) dr &= \frac{(-1)^k}{2^k} \int_0^\infty \left(\frac{1}{r} \cdot \frac{d}{dr}\right)^k e^{-\lambda r^2} (r^{2k} M_f(x,r)) dr \\ &= \frac{1}{2^k} \int_0^\infty e^{-\lambda r^2} \left[r \cdot \left(\frac{1}{r} \cdot \frac{d}{dr}\right)^k (r^{2k-1} M_f(x,r))\right] dr. \end{aligned}$$

Now have Laplace transforms on both sides, use uniqueness of the Laplace transform to find

$$\begin{aligned} u(x,t) &= \frac{\omega_n}{\pi^k 2^{k+1}} t\left(\frac{1}{t}\cdot\frac{\partial}{\partial t}\right)^k \left[t^{n-2} M_f(x,t)\right] \\ &= \frac{\omega_n}{\pi^k 2^{k+1}} t\left(\frac{1}{t}\cdot\frac{\partial}{\partial t}\right)^{(n-3)/2} \left[t^{n-2} M_f(x,t)\right] \end{aligned}$$

2 Distributions

Let $U\subset\mathbb{R}^n$ be open.

Definition 2.1. *The set of* test functions $D(U)$ *is the set of* $C_c^\infty(U)$ *(*C^∞ *with compact support). The topology on this set is given by* $\varphi_k\to\varphi$ *in* $D(U)$ *iff*

a) *there is a fixed compact set* $F\subset U$ *such that* $\operatorname{supp}\varphi_k\subset F$ *for every* k

b) $\sup_F|\partial^\alpha\varphi_k-\partial^\alpha\varphi|\to 0$ *for every multi-index* α.

Definition 2.2. *A* distribution *is a continuous linear functional on* $D(U)$. *We write* $L\in D'(U)$ *and* (L,φ).

Definition 2.3. *[Convergence on* D'*] A sequence* $L_k\overset{D'}{\to}L$ *iff* $(L_k,\varphi)\to(L,\varphi)$ *for every test function* φ.

Example 2.4. $L^p_{\mathrm{loc}}(U):=\{f:U\to\mathbb{R}: f \text{ measurable}, \int_{U'}|f|^p dx<\infty\,\forall U'\subset\subset U\}$.
An example of this is $U=\mathbb{R}$ and $f(x)=e^{x^2}$.
We associate to every $f\in L^p_{\mathrm{loc}}(U)$ a distribution L_f (here: $1\leqslant p\leqslant\infty$).

$$(L_f,\varphi):=\int_U f(x)\varphi(x)dx.$$

Suppose $\varphi_k\overset{D}{\to}\varphi$. Need to check

$$(L_f,\varphi_k)\to(L_f,\varphi).$$

Since $\operatorname{supp}\varphi_k\subset F\subset\subset U$, we have

$$\begin{aligned} \left|(L_f,\varphi_k)-(L_f,\varphi)\right| &= \left|\int_F f(x)(\varphi_k-\varphi(x))dx\right| \\ &\leqslant \underbrace{\left(\int_F|f(x)|dx\right)}_{\text{bounded}}\underbrace{\sup_F|\varphi_k-\varphi|}_{\to 0}. \end{aligned}$$

If $q>p$,

$$\int_F|f(x)|^p dx \leqslant \left(\int_F 1\,dx\right)^{1-p/q}\left(\int_F|f(x)|^q\right)^{1/q}.$$

Thus, $L^q_{\mathrm{loc}}(U)\subset L^p_{\mathrm{loc}}(U)$ for every $p\leqslant q$. (*Note:* This is *not* true for $L^p(U)$.)

Example 2.5. If μ is a *Radon measure* on U, then we can define

$$(L_\mu,\varphi)=\int_U\varphi(x)\mu(dx).$$

Example 2.6. If $\mu=\delta_y$,

$$(L_\mu,\varphi)=\varphi(y).$$

Definition 2.7. *If* L *is a distribution, we define* $\partial^\alpha L$ *for every multi-index* α *by*

$$(\partial^\alpha L,\varphi):=(-1)^{|\alpha|}(L,\partial^\alpha\varphi).$$

This definition is motivated through integration by parts, noting that the boundary terms do not matter since we are on a bounded domain.

Example 2.8. If L is generated by δ_0,

$$(\partial^\alpha L, \varphi) = (-1)^\alpha \partial^\alpha \varphi(0).$$

Theorem 2.9. *$\partial^\alpha: D' \to D'$ is continuous. That is, if $L_k \xrightarrow{D} L$, then $\partial^\alpha L_k \xrightarrow{D} \partial^\alpha L$.*

Proof. Fix $\varphi \in D(U)$. Consider

$$\begin{array}{ccc} (\partial^\alpha L_k, \varphi) & \to & (\partial^\alpha L, \varphi) \\ \| & & \| \\ (-1)^\alpha (L_k, \partial^\alpha \varphi) & \to & (-1)^\alpha (L, \partial^\alpha \varphi). \end{array}$$

□

Definition 2.10. *Suppose P is a partial differential operator of order N, that is*

$$P = \sum_{|\alpha| \leqslant N} c_\alpha(x) \partial^\alpha$$

with $c_\alpha \in C^\infty(U)$.

Example 2.11. $P = \Delta$ is an operator of order 2. $P = \partial_t - \Delta$. $P = \partial_t^2 - c^2 \Delta$.

Fundamental solution for Δ:

$$\Delta K(x - y) = \delta_y \quad \text{in } D'.$$

All this means is for every $\varphi \in D$

$$\int_U \Delta K(x - y) \varphi(x) dx = \int_U \varphi(x) \delta_y(dx) = \varphi(y).$$

Definition 2.12. *We say that u solves $Pu = 0$ in D' iff*

$$\left(u, P^\dagger \varphi\right) = 0$$

for every test function φ. Here, $P^\dagger$ is the adjoint operator obtained through integration by parts: If $c_\alpha(x) = c_\alpha$ independent of x, then

$$P^\dagger = \sum_{|\alpha| \leqslant N} (-1)^{|\alpha|} c_\alpha \partial^\alpha.$$

Example 2.13. $P = \partial_t - D \Rightarrow P^\dagger = -\partial_t - \Delta$.

Example 2.14. More nontrivial examples of distributions:

1. *Cauchy Principal Value (PV) on $\mathbb{R}$:*

$$(L, v) := \lim_{\varepsilon \to 0} \int_{|x| \geqslant \varepsilon} \varphi(x) dx.$$

2. $U = (0, 1)$

$$(L, \varphi) = \sum_{k=1}^{\infty} \left(\frac{d^k}{dx^k} \varphi\right) \left(\frac{1}{k}\right),$$

which is well-defined because φ has compact support.

Uniform convergence in topology?

2.1 The Schwartz Class

Definition 2.15. *$\mathcal{S}(\mathbb{R}^n)$ Set $\varphi \in C^\infty(\mathbb{R}^n)$ with rapid decay:*

$$\|\varphi\|_{\alpha, \beta} := \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty$$

for all multiindices α, β. Topology on this class: $\varphi_k \to \varphi$ on $\mathcal{S}(\mathbb{R}^n)$ iff $\|\varphi_k - \varphi\|_{\alpha,\beta} \to 0$ for all α, β.

Example 2.16. If $\varphi \in D(\mathbb{R}^n)$ then $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $\varphi_k \to \varphi$ in $D(\mathbb{R}^n) \Rightarrow \varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$.

Example 2.17. $\varphi(x) = e^{-|x|^2}$ is in $\mathcal{S}(\mathbb{R}^n)$, but not in $D(\mathbb{R}^n)$.

$$\partial^\beta \varphi(x) = \underbrace{P_\beta(x)}_{\text{Polynomial}} e^{-|x|^2},$$

so $\|x^\alpha \partial^\beta \varphi(x)\|_{L^\infty(\mathbb{R}^n)} < \infty$.

Example 2.18. $e^{-(1+|x|^2)^\varepsilon} \in \mathcal{S}(\mathbb{R}^n)$ for every $\varepsilon > 0$.

Example 2.19.

$$\frac{1}{(1+|x|^2)^N} \in C^\infty,$$

but not in $\mathcal{S}(\mathbb{R}^n)$ for any N. For example,

$$\sup_x \left| \frac{x^\alpha}{(1+|x|^2)^N} \right| = \infty$$

if $\alpha = (3N, 0, \ldots, 0)$.

We can define a *metric* on $\mathcal{S}(\mathbb{R}^n)$:

$$\rho(\varphi, \psi) = \sum_{k=0}^\infty \frac{1}{2^k} \sum_{|\alpha|+|\beta|=k} \frac{\|\varphi - \psi\|_{\alpha,\beta}}{1 + \|\varphi - \psi\|_{\alpha,\beta}}.$$

Claim: $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R}^n) \Leftrightarrow \rho(\varphi_k, \varphi) \to 0$.

Theorem 2.20. *$\mathcal{S}(\mathbb{R}^n)$ is a complete metric space.*

Proof. Arzelà-Ascoli. □

2.2 Fourier Transform

Motivation: For the wave equation, we find formally that

$$\mathcal{F}K_t = \frac{\sin c|\xi|t}{c|\xi|}.$$

Definition 2.21. *The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ is given by*

$$(\mathcal{F}\varphi)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x) dx.$$

For brevity, also let $\hat{\varphi}(\xi) = (\mathcal{F}\varphi)(\xi)$.

Theorem 2.22. *$\mathcal{F}$ is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$, and $\mathcal{F}\mathcal{F}^* = \mathrm{Id}$, where*

$$(\mathcal{F}^*\varphi)(\xi) = (\mathcal{F}\varphi)(-\xi).$$

2.2.1 Basic Estimates

$$\begin{aligned} |\hat{\varphi}(\xi)| &\leqslant \frac{1}{(2\pi)^{n/2}} \int |\varphi(x)| \mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1+|x|)^{n+1} \frac{|\varphi(x)|}{(1+|x|)^{n+1}} \mathrm{d}x \\ &\leqslant C \|(1+|x|)^{n+1} \varphi(x)\|_\infty < \infty. \end{aligned}$$

Also,

$$\begin{aligned}\partial_\xi^\beta \hat{\varphi}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_\xi^\beta e^{-ix\cdot\beta} \varphi(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (-i\,x)^\beta e^{-ix\cdot\beta} \varphi(x) \mathrm{d}x \\ \Rightarrow \|\partial_\xi^\beta \hat{\varphi}(\xi)\|_{L^\infty} &\leqslant C\|(1+|x|)^{n+1} x^\beta \varphi\|_{L^\infty}.\end{aligned}$$

Thus show $\hat{\varphi} \in C^\infty(\mathbb{R}^n)$:

$$\begin{aligned}(-i\xi)^\alpha \hat{\varphi}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (-i\xi)^\alpha e^{-ix\cdot\xi} \varphi(x) \mathrm{d}x \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \partial_x^\alpha (e^{-ix\cdot\xi}) \varphi(x) \mathrm{d}x \\ &= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \partial_x^\alpha \varphi(x) \mathrm{d}x \\ \Rightarrow \|\xi^\alpha \hat{\varphi}(\xi)\|_{L^\infty} &\leqslant C\|(1+|x|)^{n+1} \partial_x^\alpha \varphi\|_{L^\infty}.\end{aligned}$$

Combine both estimates to find

$$\|\hat{\varphi}\|_{\alpha,\beta} = \|\xi^\alpha \partial_\xi^\beta \hat{\varphi}\|_{L^\infty} \leqslant C\|(1+|x|)^{n+1} x^\beta \partial_x^\alpha \varphi\|_{L^\infty}.$$

Example 2.23. If $\varphi(x) = e^{-|x|^2/2}$. Then $\hat{\varphi}(\xi) = e^{-|\xi|^2/2}$. $\mathcal{F}\varphi = \varphi$.

2.2.2 Symmetries and the Fourier Transform

1. *Dilation:* $(\sigma_\lambda \varphi)(x) = \varphi(x/\lambda)$.

$$\mathcal{F}(\varphi(x/\lambda))(\xi) = \frac{\lambda^n}{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x/\lambda) \mathrm{d}(x/\lambda) = \lambda^n (\mathcal{F}\varphi)(\xi\lambda).$$

   Thus $\widehat{\sigma_\lambda \varphi} = \lambda^n \sigma_{1/\lambda} \hat{\varphi}$.

2. *Translation* $\tau_h \varphi(x) = \varphi(x-h)$ for $h \in \mathbb{R}^n$. $\mathcal{F}(\tau_h \varphi)(\xi) = e^{-ih\cdot\xi} \hat{\varphi}(\xi)$.

2.2.3 Inversion Formula

For every $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{\varphi}(\xi) \mathrm{d}\xi.$$

$\varphi(x) = \mathcal{F}^* \hat{\varphi} = (\mathcal{R}\mathcal{F})\hat{\varphi}$, where $(\mathcal{R}\varphi)(x) = \varphi(-x)$.

Proof. (of Schwartz's Theorem) Show $\mathcal{F}^*\mathcal{F} e^{-|x|^2/2} = e^{-|x|^2/2}$.

Extend to dilations and translations. Thus find $\mathcal{F}^*\mathcal{F} = \mathrm{Id}$ on $\mathcal{S}$, because it is so on a dense subset. $\mathcal{F}$ is 1-1, $\mathcal{F}^*$ is onto $\Rightarrow$ but $\mathcal{F}^* = \mathcal{R}\mathcal{F}$, so the claim is proven. □

Theorem 2.24. *$\mathcal{F}$ defines a continuous linear operator from $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, with*

$$\|\hat{f}\|_{L^\infty} \leqslant \frac{1}{(2\pi)^n} \|f\|_{L^1}.$$

Theorem 2.25. *$\mathcal{F}$ defines an isometry of $L^2(\mathbb{R}^n)$.*

Theorem 2.26. *$\mathcal{F}$ defines a continuous linear operator from $L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ with $1 \leqslant p \leqslant 2$ and*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Ideas:

- Show $\mathcal{S}(\mathbb{R}^n)$ dense in $L^p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$.

- Extend $\mathcal{F}$ from $\mathcal{S}$ to L^p.

Proposition 2.27. *$C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Take a function

$$\eta_N(x) := \begin{cases} 1 & |x| \leqslant N-1, \\ 0 & |x| \geqslant N+1. \end{cases}$$

Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, consider $\varphi_N := \varphi \eta_N$.

$$\partial^\alpha \varphi_N = \partial^\alpha(\varphi \eta_N) = \sum_{|\alpha'| \leqslant |\alpha|} \partial^{\alpha'} \varphi \partial^{\alpha - \alpha'} \eta_N.$$

So $\|x^\beta \partial^\alpha \varphi_N\|_{L^\infty} < \infty$. □

Theorem 2.28. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.*

Proof. *By Mollification.* Choose $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp}(\eta) \subset B(0,1)$ and

$$\int_{\mathbb{R}^n} \eta(x) \mathrm{d}x = 1.$$

For any n, define $\eta_N(x) = N^n \eta(Nx)$. Then

$$\int_{\mathbb{R}^n} \eta_N(x) \mathrm{d}x = 1.$$

To show:

$$f * \eta_N \overset{L^p}{\rightarrow} f$$

for any $f \in L^p(\mathbb{R}^n)$.

Step 1: Suppose $f(x) = \mathbf{1}_Q(x)$ for a rectangle Q. In this case, we know $\eta_N * f = f$ at any x with $\text{dist}(x, \partial Q) \geqslant 1/N$. Therefore, $\eta_N * f \rightarrow f$ a.e. as $N \rightarrow \infty$.

$$\int_{\mathbb{R}^n} |\eta_N * f(x) - f(x)|^p \mathrm{d}x \rightarrow 0$$

by Dominated Convergence.

(*Aside: Density of C_c^∞ in $\mathcal{S}(\mathbb{R}^n)$.* (Relation to Proposition 2.27?) Given $\varphi \in \mathcal{S}(\mathbb{R}^n)$, consider $\varphi_N := \varphi \eta_N$. We have $\|\varphi_N - \varphi\|_{\alpha,\beta} \rightarrow 0$ for every α, β. In particular, we have

$$\|(|x|^{n+1} + 1)(\varphi_n - \varphi)\|_{L^\infty} \rightarrow 0.$$

$$\int_{\mathbb{R}^n} |\varphi_n - \varphi| \mathrm{d}x = \int_{\mathbb{R}} \frac{1 + |x|^{n+1}}{(1+x)^{n+1}} |\varphi_n - \varphi| \mathrm{d}x \leqslant \left(\int_{\mathbb{R}} \frac{1}{1 + |x|^{n+1}} \mathrm{d}x \right) \ldots ?$$

End aside.)

Step 2: Step functions are dense in $L^p(\mathbb{R}^n)$ for $1 \leqslant p < \infty$.

Step 3: "Maximal inequality", i.e.

$$\|f * \eta_N\|_{L^p} \leqslant C \|f\|_{L^p},$$

which we obtain by Young's inequality.

$$\begin{aligned} \|f * \eta_N\|_{L^p} &\leqslant C_p \|\eta_N\|_{L^1} \|f\|_{L^p} \\ &= C_p \|\eta\|_{L^1} \|f\|_{L^p}, \end{aligned}$$

where the constant depends on η, but not on N.

Step 4: Suppose $f \in L^p(\mathbb{R}^n)$. Pick g to be a step function such that $\|f - g\|_{L^p} < \varepsilon$ for $1 \leqslant p < \infty$. Then

$$\begin{aligned} \|f * \eta_N - f\|_{L^p} &\leqslant \|f * \eta_N - g * \eta_N\|_{L^p} + \|g * \eta_N - g\|_{L^p} + \|f - g\|_{L^p} \\ &\leqslant (C_p \|\eta\|_{L^1} + 1) \|f - g\|_{L^p} + \|g * \eta_N - g\|_{L^p}. \end{aligned}$$

□

Onwards to prove the L^2 isometry, we define

$$(f, g)_{L^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} f(x) \overline{g(x)} \mathrm{d}x.$$

Proposition 2.29. (Plancherel) *Suppose $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$(\mathcal{F}f, \mathcal{F}g)_{L^2(\mathbb{R}^n)} = (f, g)_{L^2(\mathbb{R}^n)}.$$

Proof.

$$\begin{aligned}(\mathcal{F}f, \mathcal{F}g)_{L^2(\mathbb{R}^n)} &\overset{\text{Definition}}{=} \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \bar{g}(x)\left(\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\bar{f}(\xi)\mathrm{d}\xi\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} f(x)\bar{g}(x)dx.\end{aligned}$$

□

Definition 2.30. *$\mathcal{F}: L^1(\mathbb{R}^n) \to \dot{C}(\mathbb{R}^n)$ is the extension of $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, where*

$$\dot{C}(\mathbb{R}^n) := \{h: \mathbb{R}^n \to \mathbb{R} \text{ such that } h(x) \to 0 \text{ as } |x| \to \infty\}.$$

Proposition 2.31. *This extension is well-defined.*

Proof. Suppose

$$\begin{aligned}\varphi_k &\overset{L^1}{\to} f, \\ \psi_k &\overset{L^1}{\to} f.\end{aligned}$$

Then $\|\mathcal{F}\varphi_k - \mathcal{F}\psi_k\| \to 0$:

$$\begin{aligned}|(\hat{\varphi}_k - \hat{\psi}_k)(\xi)| &= \frac{1}{(2\pi)^{n/2}}\left|\int_{\mathbb{R}^n} e^{-ix\cdot\xi}(\varphi_k - \psi_k)\right| \\ &\leqslant \frac{1}{(2\pi)^{n/2}}\|\varphi_k - \psi_k\|_{L^1} \\ &\leqslant \frac{1}{(2\pi)^{n/2}}[\|\varphi_k - f\|_{L^1} + \|f - \psi_k\|_{L^1}] \to 0.\end{aligned}$$

□

Warning: There is something to be proved for $L^2(\mathbb{R}^n)$ because

$$\frac{1}{(2\pi)^{n/2}}\int e^{-ix\cdot\xi}f(x)\mathrm{d}x$$

is *not* defined when $f \in L^2(\mathbb{R}^n)$. However $\mathcal{F}f$ in the sense of L^2-lim $\mathcal{F}\varphi_N$ where $\varphi_N \in \mathcal{S}(\mathbb{R}^n) \to f$ in L^2.

We had proven

$$\begin{aligned}\left\|\hat{f}\right\|_{L^\infty} &\leqslant \frac{1}{(2\pi)^{n/2}}\|f\|_{L^1}, \\ \left\|\hat{f}\right\|_{L^2} &= \|f\|_{L^2}.\end{aligned}$$

Definition 2.32. *A linear operator $K: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is of* type (r, s) *if*

$$\|K\varphi\|_{L^s} \leqslant C(r, s)\|\varphi\|_{L^r}.$$

Example 2.33. $\mathcal{F}$ is of type $(1, \infty)$ and $(2, 2)$.

Theorem 2.34. (Riesz-Thorin Convexity Theorem) *Suppose K is of type (r_i, s_i) for $i = 0, 1$. Then K is of type (r, s) where*

$$\begin{aligned}\frac{1}{r} &= \frac{\theta}{r_0} + \frac{1-\theta}{r_1}, \\ \frac{1}{s} &= \frac{\theta}{s_0} + \frac{1-\theta}{s_1}\end{aligned}$$

for $0 \leqslant \theta \leqslant 1$*. Moreover,*

$$C(r,s) \leqslant C_0^{\theta} C_1^{1-\theta}.$$

Proof. Yosida/Hadamard's 3-circle theorem (maximum principle). □

Corollary 2.35. $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ *has a unique extension* $\mathcal{F}\colon L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ *where* $1 \leqslant p \leqslant 2$ *and* $1/p' + 1/p = 1$*.*

Summary:

- $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$ isomorphism
- $\mathcal{F}\colon L^1 \to \dot{C}$ (either by extension or directly) not an isomorphism
- $\mathcal{F}\colon L^2 \to L^2$ (by extension) isomorphism
- $\mathcal{F}\colon L^p \to L^{p'}$ (by interpolation)

Definition 2.36. $\mathcal{S}'(\mathbb{R}^n)$ *is the space of continuous linear functionals on* $\mathcal{S}(\mathbb{R}^n)$*, called the space of* tempered distributions*. Its topology is given by* $L_k \to L$ *in* $\mathcal{S}'$ *iff*

$$(L_k, \varphi) \to (L, \varphi)$$

for all $\varphi \in \mathcal{S}$*.*

Altogether, we have $D \subset \mathcal{S} \subset \mathcal{S}' \subset D'$.

Example 2.37.

1. Suppose $f \in L^1$. Define a tempered distribution

   $$(f, \varphi) := \int_{\mathbb{R}^n} f\varphi,$$

   which is obviously continuous.

2. (A *non*-example) If $f(x) = e^{|x|^2}$, then $f \in L^1_{\text{loc}}$, so it defines a distribution, but not a *tempered* distribution.

3. $f(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$, but

   $$\int_{\mathbb{R}^n} f\varphi = \infty.$$

4. If f is such that

   $$\left\| (1+|x|^2)^{-M} f \right\|_{L^1} < \infty$$

   for some M, then $f \in \mathcal{S}'$.

   Proof. $$|(f,\varphi)| = \left| \int f\varphi \right| \leqslant \left\| (1+|x|^2)^{-M} f \right\|_{L^1} \left\| (1+|x|^2)^{M} \varphi \right\|_{L^\infty}.$$

   □

Proposition 2.38. *Suppose* $L \in \mathcal{S}'$*. Then there exists* $C > 0$*,* $N \in \mathbb{N}$ *such that*

$$|(L, \varphi)| \leqslant C \|\varphi\|_N \tag{2.1}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$*, where*

$$\|\varphi\|_N = \sum_{|\alpha|,|\beta| \leqslant N} \left\| x^\alpha \partial^\beta \varphi \right\|_{L^\infty}.$$

Corollary 2.39. *A distribution* $L \in D'$ *defines a tempered distribution* $\Leftrightarrow$ *there exist* C*,* N *such that (2.1) holds for* $\varphi \in \mathcal{S}(\mathbb{R}^n)$*.*

Proof. Suppose (2.1) is not true. Then there exist φ_k, N_k such that

$$|(L, \varphi_k)| > k \|\varphi_k\|_{N_k}.$$

Let

$$\psi_k := \frac{\varphi_k}{\|\varphi_k\|_{N_k}} \cdot \frac{1}{k}.$$

Then

$$\|\psi_k\|_{N_k} = \frac{1}{k} \to 0.$$

But $|(L, \psi_k)| > 1$. But $\psi_k \to 0$ in $\mathcal{S}(\mathbb{R}^n) \Rightarrow L$ not continuous. □

Definition 2.40. *If $K: \mathcal{S} \to \mathcal{S}$ is linear, continuous, then the* transpose *of K is the linear operator such that for every $L \in \mathcal{S}'$*

$$(L, K\varphi) = \left(K^t L, \varphi\right).$$

Theorem 2.41. *a) $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{S}'(\mathbb{R}^n)$.*

b) $D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$.

Proof. Mollification, but first verify some properties. Fix $\eta \in D(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \eta = 1.$$

Let $\eta_m(x) = m^n \eta(m\, x)$. We want to say $\eta_m * L$ is a C^∞ function for a distribution L.

Definition 2.42. *$L \in D'(\mathbb{R}^n)$, $\eta \in D(\mathbb{R}^n)$, $\eta * L$ is the distribution defined by*

$$(\eta * L, \varphi) = (L, (R\eta) * \varphi),$$

where $R\eta(x) = \eta(-x)$. If L were a function f,

$$\begin{aligned} (\eta * L, \varphi) &= \int_{\mathbb{R}^n} (\eta * f)(x)\varphi(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \eta(x-y) f(y)\mathrm{d}y\varphi(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \eta(x-y)\varphi(x)\mathrm{d}x \right) f(y) \\ &= \int_{\mathbb{R}^n} (R\eta * \varphi)(y) f(y)\mathrm{d}y. \end{aligned}$$

Theorem 2.43. *$D(\mathbb{R}^n)$ is dense in $D'(\mathbb{R}^n)$. That is, if L is a distribution, then there exists a sequence of $L_k \in D$ such that $L_k \to L$ in D'.*

Proof. By 1) Mollification and 2) Truncation.

Proposition 2.44. *$L * \eta$ is a C^∞ function. More precisely, $L * \eta$ is equivalent to the distribution defined by the C^∞ function*

$$\gamma(x) = (L, \tau_x(R\eta)),$$

where $\tau_x f(y) = f(y - x)$.

Proof. 1) $\gamma: \mathbb{R}^n \to \mathbb{R}$ is clear.

2) γ is continuous: If $x_k \to x$, then $\gamma(x_k) \to \gamma(x)$. Check

$$\gamma(x_k) = (L, \tau_{x_k}(R\eta)).$$

And $\tau_{x_k}(R\eta) \to \tau_x(R\eta)$ *in* D.

- We can choose F s.t. $\mathrm{supp}(\tau_{x_k}(R\eta)) \subset F$ for all k.
- $R\eta(y - x_k) \to R\eta(y - x)$,
- $\partial^\alpha(R\eta)(y - x_k) \to \partial^\alpha R\eta(y - x)$,

where the last two properties hold uniformly on F.

3) $\gamma \in C^1$: Use finite differences. Consider

$$\frac{\gamma(x + h\, e_j) - \gamma(x)}{h} = \left(L, \frac{\tau_{x+he_j}(R\eta) - \tau_x(R\eta)}{h} \right).$$

Observe that

$$\frac{1}{h}\left[\tau_{x+he_j}(R\eta) - \tau_x(R\eta)\right] \to \tau_x(\partial_{x_j} R\eta)$$

in D.

4) $\gamma \in C^\infty$: Induction.

5) Show that $L * \eta \overset{D'}{=} \gamma$. That is

$$(L * \eta, \varphi) \overset{\text{Def}}{=} (L, R\eta * \varphi) \overset{?}{=} \int_{\mathbb{R}^n} \gamma(x)\varphi(x)\mathrm{d}x.$$

$$\begin{aligned} \int_{\mathbb{R}^n} \gamma(x)\varphi(x)\mathrm{d}x &= \lim_{h\to 0} h^{-n} \sum_{y \in h\mathbb{Z}^n} \gamma(y)\varphi(y) \\ &= \lim_{h\to 0} h^{-n} \sum_{y \in h\mathbb{Z}^n} (L, \tau_y(R\eta))\varphi(y) \\ &= \lim_{h\to 0} \left(L, h^{-n} \sum_{y \in h\mathbb{Z}^n} \tau_y(R\eta)\varphi(y) \right). \end{aligned}$$

Show that the Riemann sum

$$h^{-n} \sum_{y \in h\mathbb{Z}^n} \tau_y(R\eta)\varphi(y) \to R\eta * \varphi$$

in D. □

Operations with $*$ *:*

1. $\eta * L := L * \eta$.
2. $\partial^\alpha (L * \eta) \overset{D'}{=} \partial^\alpha L * \eta \overset{D'}{=} L * \partial^\alpha \eta$.

Proof of Theorem: Fix $L \in D'(\mathbb{R}^n)$. Fix $\eta \in D(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \eta(x)\mathrm{d}x = 1$. Let $\eta_m(x) = m^n \eta(m\, x)$. Then

$$\int_{\mathbb{R}^n} \eta_m(x)\,\mathrm{d}x = 1.$$

We know from our proposition from that $\eta_m * L$ is C^∞. Consider the cutoff function

$$\chi_m(x) := \begin{cases} 1 & |x| \leqslant m, \\ 0 & |x| > m. \end{cases}$$

Consider $L_m = \chi_m(\eta_m * L)$. $L_m \in D(\mathbb{R}^n)$.

$$\partial^\alpha(\chi_m \gamma_m) = \sum \binom{\alpha}{\beta} \partial^{\alpha-\beta}\chi_m \partial^{\alpha-\beta}\gamma_m$$

Claim: $L_m \to L$ in D'.

$$\begin{aligned} (L_m, \varphi) &= (\chi_m(\eta_m * L), \varphi) = (\eta_m * L, \chi_m \varphi) \\ &\overset{\text{Def}}{=} (L, (R\eta_m) * (\chi_m \varphi)). \end{aligned}$$

Finally, show

$$(R\eta_m) * \varphi \overset{m \text{ large}}{=} (R\eta_m) * \chi_m \varphi \to \varphi \quad \text{in } D'.$$

□ □

Definition 2.45. *Suppose* $K: \mathcal{S} \to \mathcal{S}$ *is linear. We define* $K^t: \mathcal{S}' \to \mathcal{S}'$ *as the linear operator*

$$\left(K^t L, \varphi\right) := (L, K\varphi).$$

Proposition 2.46. *Suppose $K: \mathcal{S} \to \mathcal{S}$ is linear and continuous. Suppose that $K_t|_{\mathcal{S}}$ is continuous. Then, there exists a unique, continuous extension of K^t to $\mathcal{S}'$.*

Corollary 2.47. *$\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ is continuous.*

Examples:

1. $\mathcal{F}\delta = 1/(2\pi)^{n/2}$.
2. Let $0 < \beta < n$ and $C_\beta = \Gamma((n-\beta)/2)$. Then $\mathcal{F}(C_\beta|x|^{-\beta}) = C_{n-\beta}|x|^{-(n-\beta)}$. Why we care: $\Delta u = \delta_0$. In Fourier space:

$$\begin{aligned} -|\xi|^2\hat{u} &= \frac{1}{(2\pi)^{n/2}} \\ \Rightarrow \hat{u} &= \frac{-1}{(2\pi)^{n/2}}|\xi|^{-2}. \\ \Rightarrow \mathcal{F}^{-1}\hat{u} &= \frac{-1}{(2\pi)^{n/2}}\frac{C_{n-2}}{C_2}|x|^{2-n}. \end{aligned}$$

Prove (1) and (2) by testing against Gaussians.

3 More about the Wave Equation

3.1 Duhamel's Principle

Consider constant coefficient linear PDE

$$\partial_t^m u + \partial_t^{m-1}\left(\sum_{|\alpha| \leqslant 1} c_{1,\alpha}\partial^\alpha\right)u + \partial_t^{m-2}\left(\sum_{|\alpha| \leqslant 2} c_{2,\alpha}\partial^\alpha\right)u + \cdots + \sum_{|\alpha| \leqslant m} c_{m,\alpha}\partial^\alpha u = 0.$$

Here $u: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, m is the order of the equation, $c_{m,\alpha} \in \mathbb{R}$.

Shorthand $P(D,\tau)u = 0$. Here $D = (\partial_{\alpha_1}, \ldots, \partial_{\alpha_n})$ and $\tau = \partial_t$ differentiation operators.

$$P(D,\tau) = \tau^m + \tau^{m-1}P_1(D) + \cdots + P_m(D).$$

$P_k(D) =$ polynomial in D of order $\leqslant k$.

General Problem:

$$P(D,\tau)u = \omega$$

for $x \in \mathbb{R}^n$, $t > 0$ with

$$\begin{aligned} u &= f_0 \\ \partial_t u = \tau u &= f_1 \\ \vdots &\quad \vdots \\ \partial_t^{m-1}u = \tau^{m-1}u &= f_{m-1} \end{aligned}$$

at $t = 0$.

Standard Problem:

$$P(D,\tau)u = 0$$

with

$$\begin{aligned} u &= 0 \\ \partial_t u = \tau u &= 0 \\ \vdots &\quad \vdots \\ \partial_t^{m-1}u = \tau^{m-1}u &= g \end{aligned}$$

at $t=0$. (Initial conditions). Solution of General Problem from Standard Problem. First, suppose $\omega \neq 0$ and $f_0 = f_1 = \cdots = f_{m-1} = 0$.

Consider the solution to a family of standard problems:

$$\begin{aligned}
P(D,\tau)U(x,t,s) &= 0 \quad (s \leqslant t)\\
\tau^{m-1}U(x,t,s) &= \omega(x,s) \quad (t=s)\\
\tau^k U(x,t,,s) &= 0 \quad (t=s, 0\leqslant k \leqslant m-2)
\end{aligned}$$

Consider

$$u(x,t) = \int_0^t U(x,t,s)\mathrm{d}s.$$

This gives us

$$\begin{aligned}
P(D,\tau)u(x,t) &= \int_0^t P(D,\tau)U(x,t,s)\mathrm{d}s + (\tau^{m-1} + \tau^{m-2}P_1(D) + \cdots + P_{n-1}(D))U(x,t,t)\\
&= 0 + \omega(x,t) + 0
\end{aligned}$$

as desired. Similarly, getting rid of non-standard initial conditions involves consideration of

$$\begin{aligned}
P(D,\tau) &= 0\\
u &= f_0\\
\tau u &= f_1\\
\vdots & \quad \vdots\\
\tau^{m-1}u &= f_{m-1}
\end{aligned}$$

Let u_g dentote the solution to the standard problem. Consider

$$u = u_{f_{m-1}} + (\tau + P_1(D))u_{f_{m-2}} + (\tau^2 + P_1(D)\tau + P_2(D))u_{f_{m-3}} + \cdots + (\tau^{m-1} + P_1(D)\tau^{m-2} + \cdots + P_{m-1}(D))u_{f_0}.$$

Then

$$\begin{aligned}
P(D,\tau)u &= P(D,\tau)u_{f_{m-1}} + (\tau + P_1(D))P(D,\tau)u_{f_{m-2}} + \cdots\\
&= 0
\end{aligned}$$

since $P(D,\tau)u_{f_k} = 0$ for $0 \leqslant k \leqslant m-1$. We need to check the initial conditions: At $t=0$, $\tau^l u_{f_k} = 0$, $0 \leqslant l \leqslant m-2$. Thus, all terms except the last one are 0. The last term is

$$\left[\tau^{m-1} + P_1(D)\tau^{m-2} + \cdots + P_{n-1}(D)\right]u_{f_0} = \tau^{m-1}u_{f_0} + \text{time derivatives of order } \leqslant m-2 \ (=0) = f_0.$$

3.2 Hyperbolicity and the Standard Problem

Henceforth, only consider the standard problem

$$\begin{aligned}
P(D,\tau) &= 0,\\
\tau^k u(x,0) &= 0 \quad (0 \leqslant k \leqslant m-2),\\
\tau^{m-1}u(x,0) &= g.
\end{aligned}$$

Solve by Fourier analysis:

$$\hat{u}(\xi,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x,t)\mathrm{d}x.$$

Fourier transform of the above standard problem yields

$$\begin{aligned}
P(i\xi,\tau)\hat{u} &= 0,\\
\tau^k\hat{u}(\xi,\tau) &= 0,\\
\tau^{m-1}\hat{u}(\xi,0) &= \hat{g}(\xi)
\end{aligned}$$

Fix ξ and suppose $Z(\xi,t)$ denotes the solution t_0 to the ODE

$$P(i\xi,\tau)Z(\xi,t)=0$$

with initial conditions

$$\tau^k Z(\xi,0)=0 \quad (0\leqslant k\leqslant m-2), \quad \tau^{m-1}Z(\xi,0)=1.$$

This is a constant coefficients ODE, an analytic solution for it exists for all t. Clearly, by linearity

$$\hat{u}(\xi,t)=Z(\xi,t)\hat{g}(\xi)$$

and

$$u(x,t)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}Z(\xi,t)\hat{g}(\xi)\mathrm{d}\xi.$$

We want $u\in C^m$ ("classical solution"). Problem: Need to show that $Z(\xi,t)$ does not grow too fast (=faster than a polynomial) in ξ. Formally,

$$\partial^\alpha\tau^k u(x,t) \;=\; \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}(i\xi)^\alpha\tau^k Z(\xi,t)\hat{g}(\xi)\mathrm{d}\xi.$$

Key estimate: For any $T>0$, there exists C_T, N such that

$$\max_{0\leqslant k\leqslant m}\sup_{0\leqslant\tau\leqslant T}\sup_{\xi\in\mathbb{R}^n}\left|\tau^k Z(\xi,t)\right|\leqslant C_T(1+|\xi|)^N$$

Definition 3.1. *The above standard problem is called* hyperbolic *if there exists a C^m solution for every $g\in\mathcal{S}(\mathbb{R}^n)$.*

Theorem 3.2. (Gårding's criterion) *The problem is hyperbolic iff $\exists c\in\mathbb{R}$ such that $P(i\xi,\lambda)\neq 0$ for all $\xi\in\mathbb{R}^n$ and λ with $\mathrm{Im}(\lambda)\leqslant -c$.*

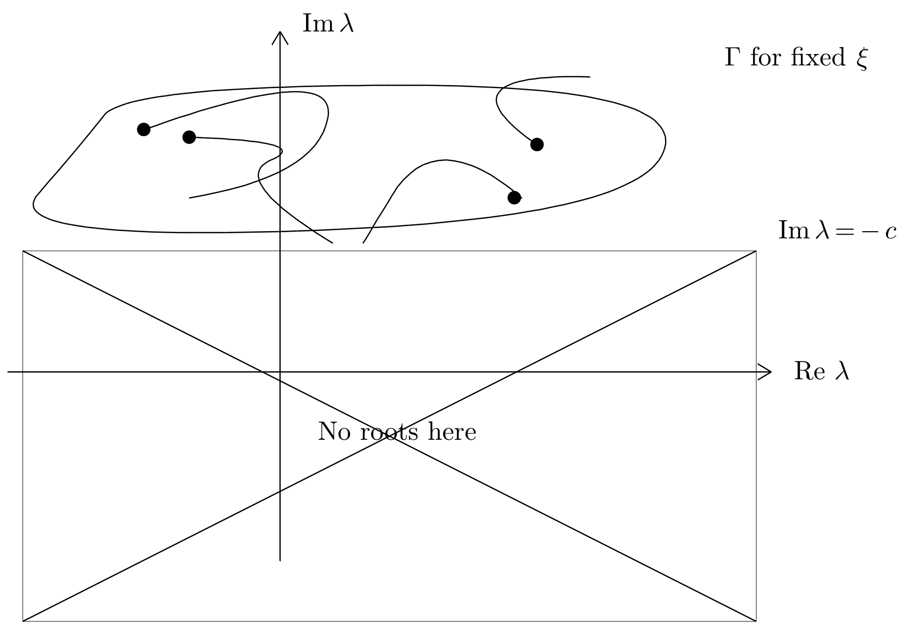

Figure 3.1. Nice cartoon.

Proof. *Cartoon:* Typical solutions to $P(i\xi,\tau)Z=0$ are of the form $Z=e^{i\lambda t}$ with $P(i\xi,i\lambda)=0$. We will only prove "$\Leftarrow$": We'll prove the estimate

$$\max_{0\leqslant k\leqslant m}\sup_{0\leqslant\tau\leqslant T}\sup_{\xi\in\mathbb{R}^n}|Z(\xi,t)|\leqslant C_T(1+|\xi|)^N$$

assuming $P(i\xi, i\lambda) \neq 0$ for $\mathrm{Im}(\lambda) \geqslant -c$. Formula for $Z(\xi, t)$:

$$Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda.$$

Claim: $P(i\xi, \tau) Z = 0$ $(t > 0)$, $\tau^k Z = 0$ $(0 \leqslant k \leqslant m-2,\ t = 0)$, $\tau^{m-1} Z = 1$ $(t = 0)$.

$$\tau^k Z = \frac{1}{2\pi} \int_\Gamma \frac{(i\lambda)^k e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda.$$

Therefore

$$\begin{aligned} P(i\xi, \tau) Z &= \frac{1}{2\pi} \int_\Gamma P(i\xi, i\lambda) \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda \\ &= \frac{1}{2\pi} \int_\Gamma e^{i\lambda t} \mathrm{d}\lambda = 0 \end{aligned}$$

by Cauchy's Theorem. Suppose $0 \leqslant k \leqslant m-2$. Let $t = 0 \Rightarrow e^{i\lambda t} = 1$.

$$\tau^k Z = \frac{1}{2\pi} \int_\Gamma \frac{(i\lambda)^k}{(i\lambda)^n \left(1 + o\left(\frac{1}{|\lambda|}\right)\right)} \mathrm{d}\lambda.$$

Suppose that Γ is the circle of radius $R \gg 1$ with center at 0. Then

$$|\tau^k Z| \leqslant \frac{1}{2\pi} \frac{R^k}{R^n \left(1 + o\left(\frac{1}{R}\right)\right)} \cdot 2\pi R = R^{k-(m-1)} \left(1 + o\left(\frac{1}{R}\right)\right) \to 0$$

if $k \leqslant m-2$. Thus, $\boxed{\tau^k Z = 0}$ for any Γ enclosing all roots.

When $k = m-1$, we have

$$\tau^{m-1} Z = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda \left(1 + \underbrace{o\left(\frac{1}{\lambda}\right)}_{\text{analytic}}\right)} \mathrm{d}\lambda = 1.$$

Step 2) Claim: Any root of $P(i\xi, i\lambda)$ staisfies

$$|\lambda(\xi)| \leqslant M(1 + |\xi|).$$

Estimate growth of roots: Suppose λ solves $P(i\xi, i\lambda) = 0$. Then

$$(i\lambda)^n + (i\lambda)^{n-1} P_1(i\xi) + \cdots + P_m(i\xi) = 0.$$

Thus,

$$-(i\lambda)^m = (i\lambda)^{m-1} P_1(i\xi) + (i\lambda)^{m-2} P_2(i\xi) + \cdots + P_m(i\xi).$$

Observe that

$$|P_k(i\xi)| \leqslant C_k (1 + |\xi|)^k \tag{3.1}$$

for every k, $1 \leqslant k \leqslant m$. Therefore,

$$|\lambda|^m \leqslant C \sum_{k=1}^m |\lambda|^{m-k} (1 + |\xi|)^k.$$

Claim: this implies:

$$|\lambda| \leqslant (1 + C)(1 + |\xi|).$$

Let

$$\theta = \frac{|\lambda|}{1 + |\xi|}.$$

Then (3.1) implies

$$\theta^m \leqslant C \sum_{k=1}^m \theta^k \Rightarrow \theta^m \leqslant \frac{\theta^m - 1}{\theta - 1} \quad (\theta \neq 1).$$

Cases:

- $\theta \leqslant 1 \Leftrightarrow |\lambda| \leqslant 1 + |\xi| \Rightarrow$ nothing to prove.
- $\theta > 1 \Rightarrow \theta^m \leqslant C\theta^m/(\theta - 1) \Rightarrow \theta \leqslant 1 + C \Rightarrow |\lambda| \leqslant (1 + C)(1 + |\xi|)$.

Step 3. Claim:

$$|\tau^k Z(\xi, t)| \leqslant M m\, e^{(1+c)t}(1 + |\xi|)^k.$$

Here M=bound from step 2, m=order of $P(D, \tau)$, c=constant in Gårding's criterion.

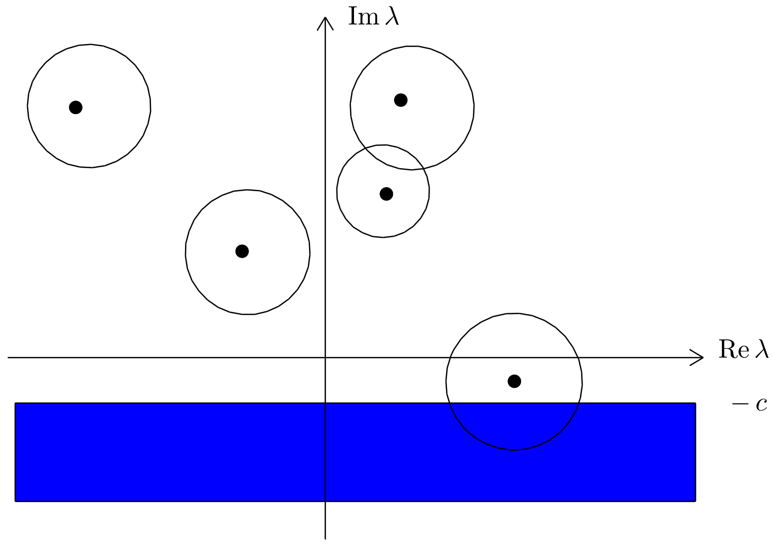

Figure 3.2. Sketch.

Fix $\xi \in \mathbb{R}^n$. Let Γ=union of circles of unit radius abound each λ_k. (wlog, no λ_k on the boundary, else consider circles of radius $1 + \varepsilon$)

$$P(i\xi, i\lambda) = i^m \prod_{k=1}^{m} (\lambda - \lambda_k(\xi)).$$

On Γ we have $|\lambda - \lambda_k(\xi)| \geqslant 1$ for all λ. Therefore $|P(i\xi, i\lambda)| \geqslant 1$ on Γ.

$$\tau^k Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{(i\lambda)^k e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda$$

Bound on $|e^{i\lambda t}|$ on Γ. we have $\mathrm{Im}(\lambda) \geqslant -c - 1$ by Gårding's assumption.

$$|e^{i\lambda t}| = e^{-(\mathrm{Im}\lambda)t} \leqslant e^{(1+c)t}.$$

Thus,

$$\begin{aligned}
|\tau^k Z(\xi, t)| &\leqslant \frac{1}{2\pi}\left(\sup_{\lambda \in \Gamma} |\lambda|^k\right) e^{(1+c)t} \underbrace{(2\pi m)}_{\text{length of }\Gamma} \\
&\leqslant m\, e^{(1+c)t}\left(\sup_l (|\lambda_l(\xi)| + 1)\right)^k \\
&\leqslant m\, e^{(1+c)t}(M(1 + |\xi|) + 1)^k
\end{aligned}$$

since each $\lambda(\xi) \leqslant M(1+|\xi|)$.

$$|\tau^k Z(\xi,t)| \leqslant C M^k m\, e^{(1+c)t}(1+|\xi|)^k.$$

Step 4. This implies that

$$\begin{aligned} |\partial^\alpha \tau^k u(x,t)| &\leqslant \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\tau^k Z(\xi,t)|\,|\xi|^\alpha \hat{g}(\xi) \mathrm{d}\xi. \\ &\leqslant \frac{C M^k e^{(1+c)t}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (1+|\xi|)^k |\xi|^\alpha |\hat{g}|(\xi) \mathrm{d}\xi < \infty \end{aligned}$$

because $\hat{g} \in \mathcal{S}$. □

Theorem 3.3. *Assume $P(D,\tau)$ satisfies Gårding's criterion. Then there exist C^∞ solutions for all $g \in \mathcal{S}(\mathbb{R}^n)$.*

For finite regularity, we only need check for $k + |\alpha| \leqslant m$. We need

$$(1+|\xi|)^m |\hat{g}(\xi)| \in L^1(\mathbb{R}^n).$$

Need for every $\varepsilon > 0$

$$(1+|\xi|)^m |\hat{g}(\xi)| \leqslant \frac{C_\varepsilon}{(1+|\xi|^{n+\varepsilon})}$$

or

$$\Big|\hat{g}(\xi)| \leqslant C_\varepsilon (1+|\xi|)^{-(m+n)-\varepsilon}.$$

m=order of $P(D,\tau)$=regularity of solution, n=space dimension.

Example 3.4. $\partial_t^2 - \Delta u = 0$. $(i\lambda)^2 - (i|\xi|)^2 = 0$, $\lambda = \pm|\xi| \rightarrow$ Growth estimate can't be improved.

Question: Is a hyperbolic equation hyperbolic in the sense that it is "wavelike" (meaning if g has compact support, $u(x,t)$ has compact support (in x) for each $t > 0$.

Theorem 3.5. (Paley-Wiener) *Suppose $g \in L^1(\mathbb{R}^n)$ with compact support. Then $\hat{g}: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is* entire.

Proof.

$$\hat{g}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{B(0,R)} e^{-ix\cdot\xi} g(x) \mathrm{d}x.$$

Formally differentiate once, then C^∞ follows. □

Theorem 3.6. *Assume Gårding's criterion (restriction on roots). Then there is a C^∞ solution to the standard problem for $g \in \mathcal{S}(\mathbb{R}^n)$.*

Example 3.7.

$$\begin{aligned} P(D,\tau)u &= u_{tt} - \Delta u \\ P(i\xi, i\lambda) &= -\lambda^2 + |\xi|^2 \end{aligned}$$

The roots are $\lambda = \pm|\xi|$, which satisfies (GC).

Example 3.8. Suppose $P(i\xi, i\lambda)$ is homogeneous

$$P(i\,s\xi, i\,s\,\lambda) = s^n P(i\xi, i\lambda)$$

for every $s \in \mathbb{R}$. (GC) holds $\Leftrightarrow$ all roots are real–otherwise, we can scale them out as far as we need to.

In general, we can write

$$P(i\xi, i\lambda) = p_{m-1}(i\xi, i\lambda) + \cdots + p_0(i\xi, i\lambda),$$

where p_k is homogeneous of degree k.

Corollary 3.9. *Suppose $P(D,\tau)$ is hyperbolic. Then all roots of $p_m(i\xi, i\lambda)$ are real for every $\xi \in \mathbb{R}^n$.*

Corollary 3.10. *Suppose the roots of p_m are real and distinct for all $\xi \in \mathbb{R}^n$. Then P is hyperbolic. (m=order of P).*

Proof. write $\xi = \rho\eta$, $\lambda = \rho\mu$. where $|\eta| = 1$, $\rho = |\xi|$.

$$P(i\xi, i\lambda) = 0 \Leftrightarrow p_m(i\eta, i\mu) + \frac{1}{\rho} p_{m-1}(i\eta, i\mu) + \cdots + \frac{1}{\rho^n} p_0(i\eta, i\mu) = 0.$$

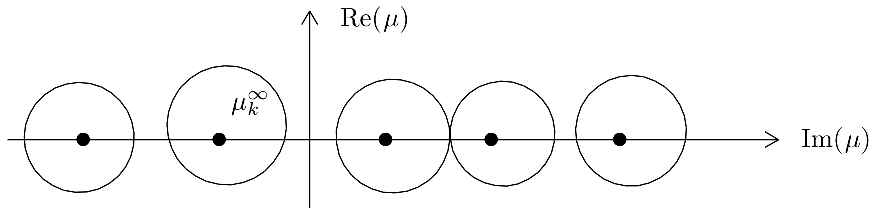

Figure 3.3. Illustrative Sketch. :-)

Use the Implicit Function Theorem to deduce that there exists $\delta > 0$ such that each μ_k^∞ perturbs $\mu_k(p)$ for $1/\rho \leqslant \delta_0$.

$$|\mu_k^\infty - \mu_k(p)| \leqslant \frac{C}{\rho}.$$

We want $f(x(\varepsilon), \varepsilon) = 0$. We know $f(x_0, 0) = 0$. The distinctness is guaranteed by the derivative condition. □

Definition 3.11. *$P(D, \tau)$ is called* strictly hyperbolic *if all $\lambda(\xi)$ are real and distinct. Also say that $p_m(D,\tau)$ is* strictly hyperbolic *if roots are real and distinct.*

Example 3.12. $u_{tt} - \Delta u = 0$ is strictly hyperbolic.

Example 3.13. (*Telegraph equation*) $u_{tt} - \Delta u + k\, u = 0$ with $k \in \mathbb{R}$. By Corollary 3.10, this equation is hyperbolic.

Theorem 3.14. *Suppose $p_m(D,\tau)$ is strictly hyperbolic. Suppose $g \in \mathcal{S}(\mathbb{R}^n)$ and* supp $g \subset B(0,a)$. *Then there exists a c_* such that*

$$\operatorname{supp} u \subset B(0, a + c_* t).$$

c_* is the largest wave speed.

Proof. (Main ingredients)

- Paley-Wiener Theorem: Suppose $g \in L^1(\mathbb{R}^n)$ and supp $g \subset B(0, a)$. Then $\hat{g}$ extends to an entire function $\mathbb{C}^n \to \mathbb{C}^n$ and

  $$|\hat{g}(\xi + i\zeta)| \leqslant \frac{\|g\|_{L^1}}{(2\pi)^{n/2}} e^{a|\zeta|}.$$

  (Proof see below)

- Heuristic:
  - Decay in $f \Rightarrow$ regularity of $\hat{f}$.
  - Regularity of $f \Rightarrow$ decay of $\hat{f}$.

- Estimates of $\operatorname{Im}(\lambda)$ for complex $\xi + i\zeta$. Use strict hyperbolicity to show

  $$\operatorname{Im}(\lambda_k) \leqslant c_*(1 + |\zeta|)$$

  for all $\zeta \in \mathbb{R}^n$.

- Plug into

$$Z(i\xi,t)=\frac{1}{2\pi}\int_{\Gamma}\frac{e^{i\lambda t}}{P(i\xi-\zeta,\lambda)}\mathrm{d}\lambda.$$

- Use

$$u(x,t)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-ix\cdot(\xi+i\zeta)}Z(\xi+i\zeta,\tau)g(\xi)\mathrm{d}\xi.$$

□

Proof. (of Paley-Wiener)

$$\begin{aligned}|\hat{g}(\xi+i\zeta)| &= \left|\frac{1}{(2\pi)^{n/2}}\int_{B(0,a)}e^{-ix\cdot(\xi+i\zeta)}g(x)\mathrm{d}x\right|\\ &\leqslant \frac{1}{(2\pi)^{n/2}}\int_{B(0,a)}|e^{-ix\cdot(\xi+i\zeta)}|\,|g(x)|\mathrm{d}x\\ &\leqslant \frac{1}{(2\pi)^{n/2}}e^{a|\zeta|}\int_{B(0,a)}|g(x)|\mathrm{d}x.\end{aligned}$$

□