256 Summary

1 High order FD

- *Finite-order finite differences*:

$$\begin{aligned} \mathcal{D}_n f(x_j) &= \frac{f_{j+n} - f_{j-n}}{2n\Delta x} \\ \frac{\mathrm{d}f}{\mathrm{d}x_j}|_{x_j} &= \sum_{n=1}^{m} \alpha_n^m \mathcal{D}_n f_j \\ \alpha_n^m &= -2(-1)^n \frac{(m!)^2}{(m-n)!(m+n)!}. \end{aligned}$$

- *Points per Wavelength*:

$$\mathrm{PPW} = \frac{2\pi}{k\Delta x} \geqslant 2$$

- *Number of passes*:

$$\nu = \frac{k\,c\,t}{2\pi}$$

- *Phase error*: Leading term of the relative error. Often

$$\mathrm{PE}(p,\nu) \sim C\nu \left(\frac{2\pi}{\mathrm{PPW}}\right)^{\mathrm{order}}.$$

- *Work per wavelength*:

$$W_m = 2m \times \mathrm{PPW} \times \frac{t}{\Delta t},$$

  where $m = \text{order}$.

- *Infinite-order finite differences*: As above with $m \to \infty$. Demand exactness for trig. polynomial e^{ilx}. Find coefficients by comparing with Fourier series for $x \mapsto x$.
  Rearranging the sum gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}|_{x_j} = \underbrace{\sum_{i=0}^{N} \frac{1}{2}(-1)^{j+i}\left[\sin\left(\frac{\pi}{N+1}(j-i)\right)\right]^{-1}}_{D_{i,j}} u_i.$$

2 Trigonometric Polynomial Approximation

Assume $u: [0, 2\pi] \to \mathbb{R}$ periodic.

- N even.
- *Spaces*:

$$\begin{aligned} \hat{B}_N &:= \mathrm{span}\{e^{inx} : |n| \leqslant N/2\} \quad N+1\text{-dim.} \\ \tilde{B}_N &:= \hat{B}_n \setminus \left\{\sin\left(\frac{N}{2}x\right)\right\} \quad N\text{-dim.} \end{aligned}$$

2.1 Continuous Expansion

- *Fourier series*:

$$\begin{aligned} \mathcal{P}_N u(x) &= \sum_{n=-\infty}^{\infty} \hat{u}_n e^{inx}, \\ \hat{u}_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx} \mathrm{d}x. \end{aligned}$$

- Special cases:
  - u real $\Rightarrow \hat{u}_{-n} = \hat{u}_n^*$,
  - u even $\Rightarrow$ only cosines,
  - u odd $\Rightarrow$ only sines.
- *Approximation*:
  - $\sum_{n=-\infty}^{\infty} |\hat{u}_n|^2 < \infty \Rightarrow \|u - \mathcal{P}_N u\|_{L^2} \to 0$.
  - $\sum_{n=-\infty}^{\infty} |\hat{u}_n| < \infty \Rightarrow \|u - \mathcal{P}_N u\|_{L^\infty} \to 0$.
- $u^{(0\ldots m-1)}$ (viewed periodically) is continuous, $u^{(m)} \in L^2 \Rightarrow |\hat{u}_n| \sim (1/n)^m$.
- *Spectral convergence*: $u \in C^\infty \Rightarrow \hat{u}_n$ decays faster than any power of n.
- $\mathcal{P}\mathcal{D} = \mathcal{D}\mathcal{P}$. Projection and differentiation commute. (start with expansion above, carry out both.)
- *Truncation error*: $\mathcal{P}_N \mathcal{L}(\mathrm{Id} - \mathcal{P}_N) = 0$.

2.1.1 Approximation Theory for the Continuous Expansion

- *Sobolev norm*:

$$\|u\|_q^2 = \sum_{m=0}^{q} \|D^m u\|_{L^2}^2 \sim \sum_{n=-\infty}^{\infty} |\hat{u}_n|^2 (1 + |n|)^{2q}.$$

- Parseval's Identity:

$$\sum_n |\hat{u}_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |u|^2.$$

- $h = 1/N$.
- $u \in H^r$:

$$\|u - \mathcal{P}_{2N} u\|_{L^2} \leqslant C h^q \left\| u^{(q)} \right\|_{L^2}.$$

  Proof: Parseval, consider tail, smuggle in an $n^{2q} \cdot \frac{1}{n^{2q}}$.
- u analytic:

$$\|u - \mathcal{P}_{2N} u\|_{L^2} \leqslant C e^{-cN} \|u\|_{L^2}$$

  Proof: $\left\| u^{(q)} \right\|_{L^2} \leqslant C q! \|u\|_{L^2}$,Stirling's Formula: $q! \sim q^q e^{-q}$, $q \sim N$.
- $u \in H^r$:

$$\|u - \mathcal{P}_{2N} u\|_{H^q} \leqslant C h^{r-q} \|u\|_{H^r}.$$

  Proof: Parseval, $(1 + |n|)^{2q} \sim \frac{(1 + |n|^{2r})}{N^{2(r-q)}}$.
- $u \in C^q$, $q > 1/2$:

$$\|u - \mathcal{P}_{2N} u\|_{L^\infty} \leqslant h^{q-1/2} \left\| u^{(q)} \right\|_{L^2}.$$

  Proof: $|u - \mathcal{P}_{2N} u|$, smuggle in n^q, CSU.
- $\mathcal{L}$ *a constant coefficient differential operator*:

$$\mathcal{L}u = \sum_{j=1}^{s} a_j \frac{\mathrm{d}^j u}{\mathrm{d}x^j}.$$

$$\|\mathcal{L}u - \mathcal{L}\mathcal{P}_{2N} u\|_{H^q} \leqslant h^{r-q-s} \|u\|_{H^r}.$$

2.2 Discrete Expansion

2.2.1 Discrete Even Expansion

- $x_j = 2\pi j/N$, $j = 0 \ldots N-1$. (N points)

- *Exactness*: Periodic case: Trapezoidal rule is Gauß quadrature.

$$u \in \hat{B}_{2N-2}: \quad \frac{1}{2\pi}\int_0^{2\pi} u(x) = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j)$$

  Proof: Evaluate geometric series.

- *Coefficients*:

$$\tilde{u}_n = \frac{1}{N\tilde{c}_n}\sum_{j=0}^{N-1} e^{-inx_j}u(x_j),$$

  where $c_n = 1 + \mathbf{1}_{n=N/2}$ to compensate for $\tilde{u}_{N/2} = \tilde{u}_{-N/2}$. $\rightarrow N$ coefficients, N quadrature points.

- *Interpolant*:

$$\begin{aligned} \mathcal{I}_N u(x) &= \sum_{|n| \leqslant N/2} \tilde{u}_n e^{inx}. \\ &= \sum_{j=0}^{N-1} g_j(x) u(x_j) \end{aligned}$$

  with

$$g_j(x) = \frac{1}{N}\sin\left(N\frac{x-x_j}{2}\right)\cot\left(\frac{x-x_j}{2}\right).$$

  - $\mathcal{I}_N: L^2 \rightarrow \tilde{B}_N$.
  - $\mathcal{I}_N u(x_j) = u(x_j)$. (rewrite sums, geometric series)

- Two different ways to differentiate: go through mode space–or don't.
- Differentiation matrix is *circulant*.
- sin $N/2$ consequences:
  - $\mathcal{I}_N \frac{\mathrm{d}}{\mathrm{d}x} \neq D\mathcal{I}_N$ ($\mathrm{d}/\mathrm{d}x: \tilde{B}_N \nrightarrow \tilde{B}_N$)
  - $D^2 \neq D^{(2)}$.
- Spatial discretization does not cause phase error deterioration.

2.2.2 Discrete Odd Expansion

- $x_j = 2\pi j/(N+1)$ $j = 0 \ldots N$. ($N+1$ points)
- *Exactness*: Periodic case: Trapezoidal rule is Gauß quadrature.

$$u \in \hat{B}_{2N}: \quad \frac{1}{2\pi}\int_0^{2\pi} u(x) = \frac{1}{N+1}\sum_{j=0}^{N} u(x_j).$$

- *Coefficients*:

$$\tilde{u}_n = \frac{1}{N+1}\sum_{j=0}^{N} u(x_j)e^{-inx_j}.$$

- *Interpolant*:

$$\begin{aligned} \mathcal{J}_N u(x) &= \sum_{|n| \leqslant N/2} \tilde{u}_n e^{inx} \\ &= \sum_{l=0}^{N} u(x_l) h_l(x) \end{aligned}$$

  with

$$h_l(x) = \frac{1}{N+1}\frac{\sin\left(\frac{N+1}{2}(x-x_l)\right)}{\sin\left(\frac{1}{2}(x-x_l)\right)} = \sum_{k=-N/2}^{N/2} e^{ik(x-x_l)}.$$

  - $\mathcal{J}_N: L^2 \rightarrow \hat{B}_N$.

  - $\mathcal{J}_N u(x_j) = u(x_j)$.
  - May also be viewed as *Lagrange trigonometric interpolant*:
  - Same differentiation matrix as ∞-order FD.
- $\mathcal{I}_N \frac{\mathrm{d}}{\mathrm{d}x} = \mathcal{D}\mathcal{I}_N$.

2.2.3 Approximation Theory for Discrete Expansions

- $u \in H^q$, $q > 1/2$:

$$\tilde{c}_n \tilde{u}_n = \hat{u}_n + \sum_{|m| \leqslant \infty, m \neq 0} \hat{u}_{n+2Nm}$$

  Proof: Substitute continuous into discrete, exchange sums because of absolute convergence, smuggle+CSU.
- *Aliasing error*:

$$\mathcal{A}_N u := \tilde{c}_n \tilde{u}_n - \hat{u}_n.$$

- $u \in H^r$, $r > 1/2$:

$$\|\mathcal{A}_N u\|_{L^2} \leqslant h^r \left\| u^{(r)} \right\|_{L^2}.$$

  Proof: smuggle, CSU.
- $u \in H^r$, $r > 1/2$:

$$\|u - \mathcal{I}_{2N} u\|_{L^2} \leqslant h^r \left\| u^{(r)} \right\|_{L^2}.$$

  Proof: Error = aliasing+truncation.
- $u \in H^r$, $r > 1/2$:

$$\|\mathcal{A}_N u\|_{H^q} \leqslant h^{r-q} \|u\|_{H^r}.$$

- $u \in H^r$, $r > 1/2$:

$$\begin{aligned} \|u - \mathcal{I}_{2N} u\|_{H^q} &\leqslant h^{r-q} \|u\|_{H^r}, \\ \|\mathcal{L}u - \mathcal{L}\mathcal{I}_{2N} u\|_{H^q} &\leqslant h^{r-q-s} \|u\|_{H^r}. \end{aligned}$$

3 Fourier Spectral Methods

Consider $u_t = \mathcal{L}u$.

3.1 Fourier Galerkin

- *Defining assumption*:

$$R_N = \partial_t u_N - \mathcal{L}u_N \perp \hat{B}_N.$$

- *Build method*: Calculate residual, project onto $\hat{B}_N$, set to zero.
  - Multiplication (for nonlinear problems) becomes convolution. (e.g. Burgers)
  - More complicated nonlinearities: no way.
  - Very efficient for linear, constant-coefficient problems with periodic BCs.

3.1.1 Stability

- $\mathcal{L}$ *semi-bounded*:

$$\mathcal{L} + \mathcal{L}^* \leqslant 2\alpha \mathrm{Id}$$

  $\Rightarrow$ stability.

- *Proving semi-boundedness*: Integrate by parts.
  Examples:
  - $\mathcal{L} = a(x)\partial_x$
  - $\mathcal{L} = \partial_x b(x)\partial_x$
- $\mathcal{L}$ *semi-bounded* $\Rightarrow$ *Fourier-Galerkin stable*.
  Proof: show $\mathcal{P}_N = \mathcal{P}_N^*$ by $(\mathcal{P}_N u, v) = (\mathcal{P}_N u, \mathcal{P}_N v)$. Then $\mathcal{L}_N = \mathcal{P}_N \mathcal{L} \mathcal{P}_N$ semi-bounded.

3.2 Fourier Collocation

- *Defining assumption*:

$$R_N|_{y_j} = 0$$

- *Optionally*: Collocation points $\{y_j\} \neq$ Quadrature points $\{x_j\}$. (we won't do that)
- *Build method*: Expand u with Lagrange interpolation polynomial. Obtain residual. Set to zero at collocation points $\rightarrow$ simply replace derivatives by application of the differentiation matrix.

3.2.1 Stability

- $\mathcal{I}_N \neq \mathcal{I}_N^*$, so Fourier Galerkin proof breaks.
- *Discrete inner product*:

$$(u, v)_N = \frac{1}{N+1} \sum_{j=0}^{N} f(x_j)\overline{g(x_j)}$$

  $\|u_N\|_N = \|u_N\|_{L^2}$ for odd expansion.
  $\|u_N\|_N \sim \|u_N\|_{L^2}$ for even expansion.
- $\mathcal{L} = a(x)u(x)$, $0 < 1/k \leqslant |a(x)| \leqslant k$:
  - $\|u_N(t)\|_N \leqslant k\|u_N(0)\|$.
    Proof: Multiply by u_N/a, obtain $(1/a)\mathrm{d}/\mathrm{d}t(\sum u^2)$. Use exactness of quad. formula, periodicity to get $\mathrm{d}/\mathrm{d}t = 0$. Exploit boundedness of a.
  - $\dot{\boldsymbol{u}} = A\ D\ \boldsymbol{u}$: Use $A^{1/2}$ as a change of variables, then bound $\boldsymbol{u} = e^{-ADt}\boldsymbol{u}_0$ by saying $A^{1/2} D A^{-1/2}$ is skew-symmetric.
    Proof remains valid for $\dot{\boldsymbol{u}} = D A\boldsymbol{u}$, $\mathcal{L} = -a(x)$, ...
- $\mathcal{L} = a(x)u(x)$ with $a(x)$ changing sign, but $|a_x|/2 \leqslant \alpha$ uniformly
  - *treat skew-symmetric form*

$$\mathcal{L}u = \frac{1}{2}a\, u_x + \frac{1}{2}(a\, u)_x - \frac{1}{2}a_x u$$

    to get $\|u_N\|_N \leqslant e^{\alpha t}\|u_0\|_N$:
    Proof: Multiply by u_N, get $\mathrm{d}/\mathrm{d}t \sum u_N^2$. Integrate (exact) by parts in the second term, only third term left over, yields bound.
  - skew-symmetric equation can be written

$$\begin{aligned} \frac{\partial u_N}{\partial t} + \frac{1}{2}\mathcal{J}_N a \partial_x u_N + \frac{1}{2}\partial_x \mathcal{J}_N[a\, u_N] - \frac{1}{2}\mathcal{J}_N(a_x u_N) &= 0, \\ \frac{\partial u_N}{\partial t} + \frac{1}{2}\mathcal{J}_N a \partial_x u_N + \frac{1}{2}\partial_x \mathcal{J}_N[a\, u_N] - \frac{1}{2}(\mathcal{J}_N \partial_x(a\, u_N) - \mathcal{J}_N a \partial_x u_N) &= 0, \\ \frac{\partial u_N}{\partial t} + \mathcal{J}_N a \partial_x u_N + \underbrace{\frac{1}{2}\partial_x \mathcal{J}_N[a\, u_N] - \frac{1}{2}\mathcal{J}_N \partial_x(a\, u_N)}_{A_N :=} &= 0 \end{aligned}$$

$$\|A_N\|_{L^2} \leqslant h^{2s-1}\left\|u_N^{(2s)}\right\|_{L^2}$$

(it's $2s-1$ because A_N contains derivatives). This motivates the...

- ...*superviscosity method*

$$\tilde{\mathcal{L}}u = \mathcal{L}u + (-1)^s \frac{\varepsilon}{N^{2s-1}} \partial_x^{2s} u_N.$$

Stable if $\varepsilon >$ some constant C.
Proof: Add A_N on both sides, integrate $(u_N, A_N)_N$ by parts, $\leqslant \left\| u_N^{(s)} \right\|_{L^2}$. Bound superviscosity term by same norm, bound for $(u, \partial_t u)_N$ involving $|a_x|$ shows up.

- Using Fourier Galerkin, see that superviscosity = filtering.

- $\mathcal{L} = b(x)\partial_x^2 u$, $b > 0$:
  - matrix method: Define $D^{(2)} = D^2$, note $D^2 \boldsymbol{u} \in \hat{B}_{N-1}$, $D_{\text{real}}^{(2)} \boldsymbol{u} \in \tilde{B}_N$, use skew-hermiticity.
  - integral method: $\partial_x^2 := \mathcal{I}_N \partial_x \mathcal{I}_N \partial_x \mathcal{I}_N$, then rewrite as integral.
- $\mathcal{L} = f(U)_x$:
  - Spectral viscosity method

$$\partial_t u_N + \partial_x \mathcal{P}_N f(u_N) = \varepsilon_N (-1)^{s+1} \partial_x^s [Q_m * \partial_x^s u_N]$$

where Q_m is a filter

  - Superspectral viscosity method

$$\partial_t u_N + \partial_x \mathcal{P}_N f(u_N) = \varepsilon_N (-1)^{s+1} \partial_x^{2s} u_N.$$

4 Orthogonal Polynomials

- $B_N := \text{span}\{x^n : 0 \leqslant n \leqslant N\}$.
- Fourier methods achieve exponential accuracy only if u is periodic.
- *Sturm-Liouville operator*:

$$\mathcal{L}\varphi = \partial_x(p\partial_x\varphi) + q\varphi = \lambda w \varphi$$

$p > 0$, $0 \leqslant q < M$, w the weight function.

- *Parseval identity*:

$$(u, u)_{L_w^2} = \sum \gamma_n \hat{u}_n^2, \quad \gamma_n = (\varphi_n, \varphi_n), \quad \hat{u}_n = \frac{1}{\gamma_n}(u, \varphi_n)_{L_w^2}.$$

- Estimate decay of $\hat{u}_n$ by plugging in eigenvalue problem, using selfadjointness of operator.
- *Singular Sturm-Liouville problem*: p vanishes at boundary.

$$\rightarrow |\hat{u}_n| \sim C \frac{1}{\lambda_n^m} \left\| \left(\frac{\mathcal{L}}{w} \right)^m u \right\|_{L_w^2}.$$

$\rightarrow$ spectral decay for C^∞ functions with zero BCs. (Regular problem: only for periodic problems, otherwise boundary causes error.)

- *Jacobi polynomials*: $P_n^{(\alpha,\beta)}$, $\alpha, \beta > -1$

$$p(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}, \quad w(x) = (1-x)^\alpha (1+x)^\beta, \quad q(x) = c\, w.$$

- *Rodrigues' formula*:

$$(1-x)^\alpha (1+x)^\beta P_n^{\alpha,\beta}(x) = \frac{1}{2^n n!} \partial_x^n (1-x)^{\alpha+n} (1+x)^{\beta+n}.$$

- *Derivative*:

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n^{(\alpha,\beta)} = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x).$$

- *Odd/Even*:

$$P_n^{(\alpha,\beta)} = (-1)^n P_n^{(\alpha,\beta)}(-x).$$

- *There are various three-term recurrence for these polynomials*, $P_0^{(\alpha,\beta)} = 1$, $P_1^{(\alpha,\beta)} = \frac{1}{2}(\alpha+\beta+2)x + (\alpha-\beta)/2$.
- *Legendre polynomials*: $\alpha = \beta = 0$, $w \equiv 1$, called P_n
- *Chebyshev polynomials*: $p = \sqrt{1-x^2}$, $q = 0$, $w = p$. $T_n = \cos(n \arccos(x))$.

$$x T_n = \frac{1}{2} T_{n-1} + T_{n+1}.$$

  Chebyshev is best approximation to x^{n+1} among polynomials of degree n.
- *Ultraspherical/Gegenbauer polynomials*: $\alpha = \beta$.
- PPW for polynomials: ~ 4. (Gegenbauer expansion, decay of the Bessel function)

5 Polynomial Expansions

- Can somewhat easily differentiate and integrate, requires three-term stuff and its inverse.
- *Gauß-Lobatto quadrature*: both endpoints part of the quadrature. Exact for B_{2N-1}.
- *Gauß-Radau quadrature*: one endpoint part of the quadrature. Exact for B_{2N}.
- *Pure Gauß quadrature*: no endpoints part of the quadrature. Exact for B_{2N+1}.
- Each different kind of polynomial has a different set of quadrature points and weights because each has a different weight function.
- Chebyshev Quadrature:

$$\begin{array}{llll} \text{GL} & \text{GR} & G & \\ x_j = -\cos\left(\frac{j}{N}\pi\right) & w_j = -\cos\left(\frac{2j}{2N+1}\pi\right) & z_j = -\cos\left(\frac{2j+1}{2N+2}\pi\right) & j = 0, \ldots, N \\ w_j = \frac{\pi}{c_j N} & v_j = \frac{\pi}{c_j} \cdot \frac{1}{2N+1} & u_j = \frac{\pi}{N+1} & \end{array}$$

  with

$$c_j = 1 + \mathbf{1}_N + \mathbf{1}_0.$$

- $[\cdot,\cdot]_w$ denotes discrete inner product, $\|\cdot\|_{N,w}$ discrete norm.
- *Discrete Gauß-Lobatto norm*: not exact for $n = N$, but equivalent.
- *Discrete Expansion*:

$$\mathcal{I}_N u(x) = \sum_{n=0}^{N} P_n^{(\alpha)}(x) \tilde{u}_n, \quad \tilde{u}_n = \frac{1}{\tilde{\gamma}_n} \sum_{j=0}^{N} u(x_j) P_n^{(\alpha)}(x_j) w_j$$

- *Quadrature points are interpolation points*.
  Proof: Plug coefficient terms into expansion, exchange sums to find

$$l_j(x) = w_j \sum_{n=0}^{N} \frac{1}{\tilde{\gamma}_n} P_n^{(\alpha)}(x) P_n^{(\alpha)}(x_j)$$

  is the Lagrange interpolation polynomial.
- Differentiation matrices are nilpotent. (Decrease in order)
- GL Differentiation matrix is centro-antisymmetric.
- $D^{(q)} = D^q$.

- *Runge phenomenon*: Wild behavior of polynomials near interval boundaries.
- $u \in C^0[-1,1]$, $\{x_j\}$ interpolation nodes. Then

$$\|u - \mathcal{I}_n u\|_\infty \leqslant |1 + \Lambda_N| \|u - p^*\|_\infty,$$

where p^* is the best-approximating polynomial and

$$\Lambda_n = \max_{[-1,1]} \lambda_n, \quad \lambda_n = \sum_{j=0}^{\mathrm{N}} l_j(x).$$

- $\Lambda_N \geqslant C \log(N+1) + C'$.
- *Cauchy interpolation remainder*:

$$u(x) - \mathcal{I}_N u(x) = \frac{u^{(N+1)}(\xi)}{(N+1)!} \prod_{j=0}^{n} (x - x_j).$$

- Grid points should cluster quadratically near the boundary.

6 Polynomial Spectral Methods & Stability

6.1 Galerkin

- *Defining assumption*: Residual orthogonal to B_N.
- *Stiffness matrix*:

$$S_{k,n} = \frac{1}{\gamma_k} \int \varphi_k \mathcal{L} \varphi_n w \mathrm{d}x.$$

Mass matrix:

$$M_{k,n} = \frac{1}{\gamma_k} \int \varphi_k \varphi_n w \mathrm{d}x,$$

positive definite because L^2-norm is a norm.

- Formulation:

$$\dot{\boldsymbol{a}} = M^{-1} S \boldsymbol{a}.$$

- Basis constructed as a linear combination of $P_n^{(\alpha)}$ to ensure BCs are kept.
- $u_t = \mathcal{L}u$. If $\mathcal{L}$ is semi-bounded ($\mathcal{L} + \mathcal{L}^* \leqslant 2\gamma \mathrm{Id}$), then the Galerkin method is stable.
- Linear hyperbolic equation well-posed in Jacobi norm for $\alpha \geqslant 0$, $\beta \leqslant 0$, but not for Chebyshev. (Consider $1 - |x|/\varepsilon$. Norm blows up, because Cheb weights blow up.)

6.2 Tau

- *Defining assumption*: Residual orthogonal to B_{N-k}, where k is the number of BCs, demand that it is zero.
- BC coefficients can be obtained once PDE-discretizing coefficients are computed.
- Mass matrix remains diagonal.
- Usable for elliptic problems, allows efficient preconditioners.
- Burgers: Product once again becomes convolution-like term.

6.3 Collocation

- *Defining assumption*: Residual zero at interpolation/quadrature nodes.

- Stability: Usual go-to-integral stuff.

6.4 Penalty Method for Boundary Conditions

- *Example*:

$$Q^-(x) = \frac{(1-x)P_N'(x)}{2P_N(-1)} = \begin{cases} 1 & x = -1, \\ 0 & x = x_j \neq -1. \end{cases}$$

$$\frac{\partial u_N}{\partial t} + a\frac{\partial u_N}{\partial x} = -\tau a\, Q^-(x)(u_N(-1) - \text{BC})$$

- Consistent because exact solution satisfies scheme exactly.
- Stable: go back to integral, gives boundary values, tweak τ to be bigger than corresponding weight.