257 Summary

TODO:

- How did we deduce TVD for the PDE?
- Lax's entropy condition: $<$ or $\leqslant$?
- Strange second integral condition in derivation of Godunov.
- Why the meshing restriction for FD?

1 Miscellanea

- An interpolation polynomial is monotone in a jump cell.
  Example: Degree-five polynomial, six points, degree-four derivative, four derivative zeros in each of the boundary cells $\Rightarrow$ none in the center jump cell.

2 Theory

- *Conservation Law*: $u_t + f(u)_x = 0$. Initial condition u_0.
- *Integral form*:
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x,t) = f(u(b,t)) - f(u(a,t)). \tag{1}$$
- *Characteristic*: Defined by
$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t),t) = u_x x' + u_t \overset{!}{=} 0,$$
  setting $x' = f'$. May cross.
- *Weak solution*:
  - (1) for almost all (a,b)
  - For any $\varphi \in C_0^1(\mathbb{R}^2)$, $t > 0$
$$-\int_0^t \int_{-\infty}^{\infty} u\varphi_t + f(u)\varphi_x \mathrm{d}x\mathrm{d}t - \int_{-\infty}^{\infty} u_0(x)\varphi(x,0)\mathrm{d}x = 0.$$

  Both definitions equivalent.
- *Rankine-Hugoniot condition*: Curve parameterized by $(x(t),t)$ separates two smooth regions.
$$s = x'(t) = \frac{[\![f]\!]}{[\![u]\!]}$$
  Proof: Split (1) at $x(t)$, carry out time derivative, observe Leibniz rule, apply conservation law.
- *Riemann problem*: Conservation law with single-jump (otherwise constant) IC.
  Rarefaction $(-1, x/t, 1)$ is a weak solution, jump is also weak solution $\Rightarrow$ non-uniqueness.
  If f is convex, the general solution
$$u(x,t) = \begin{cases} \begin{cases} u_l & x < s\,t, \\ u_r & x > s\,t, \end{cases} & u_l > u_r, \\ \begin{cases} u_l & x < f'(u_l)t, \\ (f')^{-1}(x/t) & \text{otherwise}, \\ u_r & x > f'(u_r)t, \end{cases} & u_l < u_r. \end{cases}$$

- *Vanishing viscosity method*: add u^ε_{xx} to the RHS of the conservation law, letting $\varepsilon \to 0$.
- *Entropy function*: U convex ($U'' \geqslant 0$).
- *Entropy flux*: $F'(u) = U'(u) f'(u)$.
- *Entropy condition*: (U, F) an entropy-entropy flux pair. Then u is an entropy solution iff

$$U(u)_t + F(u)_x \geqslant 0$$

weakly.
Proof: Multiply c.law by $U'(u^\varepsilon)$, gather derivatives. On RHS, write

$$U'(u^\varepsilon) u^\varepsilon_{x,x} = (U'(u^\varepsilon) u^\varepsilon_x)_x - U''(u^\varepsilon)(u^\varepsilon_x)^2 \leqslant (U'(u^\varepsilon) u^\varepsilon_x)_x.$$

Then multiply by smooth $\varphi \geqslant 0$ and integrate by parts twice. Pass to limit by DCT, RHS vanishes because u^ε is bounded–maximum principle.

- The conservation law is
  - *Genuinely nonlinear*: $f''(u) \neq 0$ uniformly,
  - *Convex*: $f''(u) > 0$ uniformly,
  - *Concave*: $f''(u) < 0$ uniformly.
- *Other Entropy conditions*:
  - *Motivation*: $x'(t)[\![U]\!] \leqslant [\![F]\!]$ by applying a Rankine-Hugoniot type argument to $U(u)_t + F(u)_x \geqslant 0$.
  - *Oleinik entropy condition*: For all $u \in [u^-, u^+]$:

$$\frac{f(u) - f(u^-)}{u - u^-} \geqslant s \geqslant \frac{f(u) - f(u^+)}{u - u^+},$$

    where s is the shock speed from Rankine-Hugoniot.
  - *Lax entropy condition*:

$$f'(u^-) \geqslant s \geqslant f'(u^+).$$

    Not sufficient for uniqueness, but necessary.
    Sufficient if $f'(u) \gtreqless 0$ uniformly. Simpler if $f'(u) > 0$:

$$f'(u^-) \geqslant f'(u^+).$$

    Since f' is $\nearrow$, we can only jump down.
    Meaning: Characteristics only go *into* shocks, never out of them.
- *L^1 contraction*: For $u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{x,x}$ with $u^\varepsilon(0, t) = u_0(x)$, we have

$$\|u^\varepsilon(\,\cdot\,, t) - v^\varepsilon(\,\cdot\,, t)\|_{L^1} \leqslant \left\|u^0 - v^0\right\|_{L^1},$$

where v^ε solves the same PDE with IC v^0.
Proof: Chop up

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} |u^\varepsilon - v^\varepsilon|$$

at the sign changes. Put in s_j as a sign function on each interval. Use Leibniz's rule, then the c.law, which can be integrated out to zero, leaving some u_x terms, which can be deduced to have the right sign.

- *Total Variation*:

$$\mathrm{TV}(u) := \sup_h \int \frac{|u(x+h) - u(x)|}{h} \mathrm{d}x$$

TVD: $\mathrm{TV}(u(\,\cdot\,, t)) \leqslant \mathrm{TV}(u^0)$. XXX WHY?

3 Numerics

- *Bad example*: Discretize $u_t = u\ u_x$ 255-style, and get a monstrosity that leaves a 1-0 shock just where it is.
- *Conservative scheme*:
$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left[\hat{f}_{j+1/2} - \hat{f}_{j-1/2}\right],$$
with
  - $\hat{f}$ consistent, i.e. $\hat{f}(u,\ldots,u) = f(u)$,
  - $\hat{f}$ locally Lipschitz.
- *Summation by parts*:
$$\sum_{j=j_1}^{j_2} a_j(b_j - b_{j-1}) = -\sum_{j=j_1}^{j_2} (a_{j+1} - a_j)b_j - a_{j_1}b_{j_1-1} + a_{j_2}b_{j_2}.$$
- *Lax-Wendroff*: If $\{u_j^n\}$ converges $(\Delta t, \Delta x \to 0)$ boundedly a.e. to a function $u \Rightarrow u$ a weak solution. Proof: Summation by parts, DCT, Conservativity.
- *Schemes*:
  - *Godunov*: Exploit finite propagation speed, solve Riemann problem on each cell, demanding that
$$\int_{I_j}\int_t (u_t + f(u)_x)\mathrm{d}x\mathrm{d}t \;=\; 0.$$
To get $\hat{f}_{j+1/2}^n$, use the exact Riemann solution at $x_{j+1/2}$.
$$\hat{f}_{j+1/2} = \begin{cases} \min_{[u_j,u_{j+1}]} f(u) & u_j < u_{j+1}, \\ \max_{[u_j,u_{j+1}]} f(u) & u_j \geqslant u_{j+1}. \end{cases}$$
  - *Lax-Friedrichs*:
$$\hat{f}_{j+1/2} = \frac{1}{2}\left[f(u_{j+1}) + f(u_j) - \alpha_{j+1/2}\Delta_+ u_j\right]$$
    - *local Lax-Friedrichs*: $\alpha_{j+1/2} = \max_{[u_j,u_{j+1}]} |f'(u)|$,
    - *global Lax-Friedrichs*: $\alpha_{j+1/2} = \max_u |f'(u)|$.
  - *Roe*:
$$\hat{f}_{j+1/2} = \begin{cases} f(u_j) & a_{j+1/2} \geqslant 0 \\ f(u_{j+1}) & a_{j+1/2} < 0 \end{cases}, \quad \text{where} \quad a_{j+1/2} = \frac{\Delta_+ f(u_j)}{\Delta_+ u_j}.$$
  - *Engquist-Osher*:
$$\begin{aligned} \hat{f}_{j+1/2} &= f^+(u_j) - f^-(u_{j+1}), \\ f^+(u) &= \int_0^u f'(u) \vee 0\mathrm{d}u + f(0), \\ f^-(u) &= \int_0^u f'(u) \wedge 0\mathrm{d}u. \end{aligned}$$
  - *Lax-Wendroff*:
    - Taylor-expand u^{n+1} in t.
    - Replace time derivatives with 2nd-order centered differences to desired order.
$$\hat{f}_{j+1/2} = \frac{1}{2}\left[f(u_j) + f(u_{j+1}) - \lambda f'(u_{j+1/2})(f(u_{j+1}) - f(u_j))\right],$$
where
$$u_{j+1/2} = \frac{u_{j+1} + u_j}{2}, \quad \lambda = \frac{\Delta t}{\Delta x}.$$

- *McCormack*: Predictor-corrector-style

$$\begin{aligned}u_j^{n+1/2} &= u_j^n - \lambda(f(u_j^n) - f(u_{j-1}^n)),\\ u_j^{n+1} &= \frac{1}{2}\Big[u_j^n + u_j^{n+1/2} + \lambda\Big[f(u_{j+1}^{n+1/2}) - f(u_j^{n+1/2})\Big]\Big].\end{aligned}$$

- *Monotone schemes*: Write $u_j^{n+1} = G(u_{j-p-1}, \dots u_{j+q})$. Monotone iff $G(\uparrow, \uparrow, \uparrow)$.
  - *For three-point schemes*: $G(u_{j-1}, u_j, u_{j+1}) = u_j - \lambda[\hat{f}(u_{j+1}, u_j) - \hat{f}(u_{j-1}, u_j)] \Rightarrow G(\uparrow, ?, \uparrow)$. $\partial_{u_j} G = 1 - \lambda(\hat{f}_1 - \hat{f}_2) \geqslant 0!$
  - L-F is monotone.

  Properties:
  - $u_j \leqslant v_j$ for all $j \Rightarrow G(u_j) \leqslant G(v_j)$
    Proof: by definition.
  - Local maximum principle

$$\min_{i \in \text{stencil}_j} u_i \leqslant G(u_j) \leqslant \max_{i \in \text{stencil}_j} u_i$$

    Proof: Define w to be $\min_{\text{stencil}}$ on the stencil and u otherwise. Then

$$\min_{\text{stencil}} = G(w) \leqslant G(u).$$

  - *Crandall/Tartar Lemma/L^1 contraction*: $\|G(u) - G(v)\|_{L^1} \leqslant \|u - v\|_{L^1}$
    Proof: Let $w := u \vee v$. Then $G(u), G(v) \leqslant G(w)$ and $G(w) - G(v) \geqslant (G(u) - G(v))^+$. Then

$$\sum (G(u) - G(v))^+ \leqslant \sum [G(w) - G(v)] \overset{\text{conservative}}{=} \sum (w - v) = \sum (u - v)^+.$$

  - *TVD*. Take $v_j = u_{j+1}$ in L^1 contraction.
- *Cell entropy inequality*: Let $U(u) = |u - c|$ and $\hat{F} = \hat{f}(c \vee u) - \hat{f}(c \wedge u)$.

$$\frac{U(u_j^{n+1}) - U(u_j^n)}{\Delta t} - \frac{\hat{F}_{j+1/2} - \hat{F}_{j-1/2}}{\Delta x} \leqslant 0$$

  Proof: Show

$$G(c \vee u_j) - G(c \wedge u_j) = |u_j^n - c| - \lambda(\hat{F}_{j+1/2} - \hat{F}_{j-1/2})$$

  by starting with the LHS. Next, $c \vee u_j^{n+1} \leqslant G(c \vee u^n)_j$ and so

$$U(u^{n+1})_j = |u_j^{n+1} - c| \leqslant G(c \vee u^n)_j - G(c \wedge u^n)_j.$$

- *Godunov's Theorem*: Montone schemes are at most first-order accurate.
  Proof: The scheme is second-order accurate for an equation with dissipation, so it can't also be second-order accurate for the original c.law.
- *TVD scheme*.
- *Monotonicity-preserving scheme*:

$$u_j^n \geqslant u_{j+1}^n \forall j \quad \Rightarrow \quad u_j^{n+1} \geqslant u_{j+1}^{n+1} \forall j.$$

- *TVD $\Rightarrow$ monotonicity-preserving*.
  Proof: Suppose it isn't. Then you can make u constant outside the relevant stencils. Reversal of order of the two values implies non-TVD.
- *Linear scheme*: Linear if applied to a linear PDE.
  Also "positive" because Monotone $\Leftrightarrow$ positive coefficients.
  Can be written

$$u_j^{n+1} = \sum_{l=-k}^{k} c_l(\lambda) u_{j-l}^n.$$

- *Linear, monotonicity-preserving* $\Rightarrow$ *monotone*.
  Proof: Consider first differences of a Heaviside jump $\Rightarrow$ all coefficients positive.
- *Linear, monotone (TVD)* $\Rightarrow$ *at most first order*.
  Proof: Plug in constant, linear term, quadratic term to obtain

$$\begin{aligned} 1 &= \sum c_l, \\ \lambda &= \sum l\, c_l, \\ \lambda^2 &= \sum l^2 c_l. \end{aligned}$$

  Then $\boldsymbol{a} := (l\sqrt{c_l})$, $\boldsymbol{b} := (\sqrt{c_l})$ and Cauchy-Schwarz (equality iff $\boldsymbol{a} = \alpha\boldsymbol{b}$).

3.1 Higher Order TVD Schemes

Assume $f'(u) \geqslant 0$ (wind from the left) for the moment.

- *General Finite Volume Framework*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_{j-1/2}}^{x_{j+1/2}} u\mathrm{d}x + f(u(x_{j+1/2})) - f(u(x_{j-1/2})) = 0,$$

  then

$$\bar{u} := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u\mathrm{d}x,$$

  so

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{u}_j = \frac{1}{\Delta x_j}\Big[\hat{f}_{j+1/2} - \hat{f}_{j-1/2}\Big]$$

  with

$$\hat{f}_{j\pm1/2} \approx f(u(x_{j\pm1/2})).$$

- *Reconstruction*:

$$\begin{aligned} u_{j+1/2}^{(\text{central})} &= \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), \\ u_{j+1/2}^{(\text{upwind})} &= \frac{1}{2}\left(3\bar{u}_j - \frac{1}{2}\bar{u}_{j-1}\right). \end{aligned}$$

  Goal is to compute polynomial such that $\frac{1}{\Delta x}\int_{I_j} p(x) = \bar{u}_j$ for some js. Then evaluate at $x_{j+1/2}$.

- *minmod*:

$$\operatorname{minmod}(a,b,c) = \begin{cases} \operatorname{argmin}\{|a|,|b|,|c|\} & \text{same sign on all,} \\ 0 & \text{otherwise.} \end{cases}$$

- *Harten's lemma*:

$$\bar{u}_{j+1} = \bar{u}_j + \lambda(C_{j+1/2}\Delta_+\bar{u}_j - D_{j-1/2}\Delta_-\bar{u}_j)$$

  is TVD if:

$$\begin{aligned} C_{j+1/2} &\geqslant 0, \\ D_{j+1/2} &\geqslant 0, \\ 1 - \lambda(C_{j+1/2} + D_{j+1/2}) &\geqslant 0. \end{aligned}$$

  Proof: Look at $\sum |\Delta_+\bar{u}_j|$, observe sum-arounds.

- *MUSCL scheme*:

$$\hat{u}_{j+1/2}^{(\text{muscl})} = \bar{u}_j + \underbrace{\operatorname{minmod}\left(u_{j+1/2}^{(\text{upwind})} - \bar{u}_j, u_{j+1/2}^{(\text{central})} - \bar{u}_j\right)}_{\tilde{u}_j :=}.$$

  Is TVD by Harten's lemma.
  Proof: Take

$$\bar{u}_j^{n+1} = \bar{u}_j - \lambda[f(\bar{u}_j + \tilde{u}_j) - f(\bar{u}_{j-1} + \tilde{u}_{j-1})] = \bar{u}_j - \lambda\big[-D_{j-1/2}\Delta_-\bar{u}_j],$$

and

$$\begin{aligned} D_{j-1/2} &= \frac{f(\bar{u}_j+\tilde{u}_j)-f(\bar{u}_{j-1}+\tilde{u}_{j-1})}{\bar{u}_j-\bar{u}_{j-1}} = f'(\xi)\frac{\bar{u}_j-\bar{u}_{j-1}+\tilde{u}_j-\tilde{u}_{j-1}}{\bar{u}_j-\bar{u}_{j-1}} \\ &= f'(\xi)\Bigg[1+\underbrace{\frac{\tilde{u}_j}{\bar{u}_j-\bar{u}_{j-1}}}_{0\leqslant\cdot\leqslant\frac{1}{2}}-\underbrace{\frac{\tilde{u}_{j-1}}{\bar{u}_j-\bar{u}_{j-1}}}_{0\leqslant\cdot\leqslant\frac{1}{2}}\Bigg]\geqslant 0 \end{aligned}$$

CFL restriction: $\lambda \max|f'| \leqslant 2/3$.

Now lift wind-from-left restriction.

- General form:

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda\Big[\hat{f}(u_{j+1/2}^-, u_{j+1/2}^+) - \hat{f}(u_{j-1/2}^-, u_{j-1/2}^+)\Big],$$

  where $\hat{f}(\uparrow,\downarrow)$ is a monotone flux.

- Now choose

$$u_{j+1/2}^{+,\text{mod}} = \bar{u}_j + \text{minmod}(u_{j+1/2}^+, \bar{u}_j - \bar{u}_{j-1}, \bar{u}_{j+1} - \bar{u}_j)$$

  etc.

- Prove TVD by

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda\Bigg[\underbrace{\hat{f}(u_{j+1/2}^-, u_{j+1/2}^+) - \hat{f}(u_{j+1/2}^-, u_{j-1/2}^+)}_{C_{j+1/2}\Delta_+\text{-term}} + \underbrace{\hat{f}(u_{j+1/2}^-, u_{j-1/2}^+) - \hat{f}(u_{j-1/2}^-, u_{j-1/2}^+)}_{D_{j-1/2}\Delta_-\text{-term}}\Bigg]$$

  using Harten, monotonicity of the flux.

- Smooth and montone region → high-order accuracy.
  Proof:

$$\begin{aligned} \tilde{u}_j &= u_x\frac{\Delta x}{2} + O(\Delta x^2) \\ \bar{u}_{j+1} - \bar{u}_j &= u_x\Delta x + O(\Delta x^2) \\ \bar{u}_j - \bar{u}_{j-1} &= u_x\Delta x + O(\Delta x^2) \end{aligned}$$

  So the high-accuracy term is half as big as the low-accuracy limiting terms in the minmod.

- *TVD schemes are at most first-order accurate near smooth extrema.* Consider extremal hump between two grid points.

- *TVB scheme*:

$$\overline{\text{minmod}}(a,b,c) := \begin{cases} a & |a| \leqslant M|\Delta x|^2, \\ \text{minmod}(a,b,c) & \text{otherwise.} \end{cases}$$

  Scheme maintains high-order accuracy, choosing $M = \frac{2}{3}|u_{x,x}|$. TVB:

$$\text{TV}(\bar{u}^{n+1}) \leqslant \text{TV}(\bar{u}^{n+1}) + C\,M\,\Delta x^2\,N \leqslant \text{TV}(\bar{u}^n) + C\Delta t.$$

- *Semidiscrete Cell Entropy Inequality*:

$$\frac{\mathrm{d}U(u_j)}{\mathrm{d}t} + \frac{1}{\Delta x}\Big[\hat{F}_{j+1/2} - \hat{F}_{j-1/2}\Big] = -\frac{1}{\Delta x}\underbrace{\Theta_j}_{\geqslant 0}.$$

  Let $U''(u) \geqslant 0$ and integrate by parts in the definition of the entropy flux F. Let

$$\hat{F}_{j+1/2} = U'(u_j)\hat{f}(u_j, u_{j+1}) - \int^{u_j} U''(u) f(u)\mathrm{d}u.$$

  Multiply the c.law by $U'(u_j)$, yielding a "junk" term Θ_j that ends up being positive, proving the CEI.

3.2 ENO/WENO

- *Newton interpolation*:

$$\begin{aligned} y[x_i] &= y_i, \\ y[x_i, x_{i+1}] &= \frac{y[x_{i+1}] - y[x_i]}{x_{i+1} - x_i}, \\ y[x_i, x_{i+1}, x_{i+2}] &= \frac{y[x_{i+1}, x_{i+2}] - y[x_i, x_{i+1}]}{x_{i+2} - x_i}, \end{aligned}$$

  then

$$p(x) = y[x_0] + y[x_0, x_1](x - x_0) + y[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots.$$

- *Interpolation* $\leftrightarrow$ *Reconstruction*: Thinking about $P = \int p$, where p is the reconstruction polynomial, yields that running sums of cell averages turn reconstruction into interpolation. Since the step $P \to p$ is first differences, i.e. undoing running sums, the reconstruction polynomial for $\bar{u}$ is the same as the interpolation polynomial for $\sum \bar{u}$.
- *ENO idea*: Progressively expand the stencil in the direction with the lowest divided differences. Un-divided differences (for a uniform mesh) may be precomputed.
- *WENO idea*: Start with a linear combination of smaller stencils that gives high-order accuracy.

$$\sum_i \alpha_i \text{stencil}_i$$

  Now weight them so that $w_i = \alpha_i + O(\Delta x^2)$ in smooth regions and $w_i = O(\Delta x^4)$. The normalize the w_i so they add up to one.

3.3 Finite Differences

- *Finite Difference Idea*: View $f(u_j)$ as cell averages of a function h. Then

$$f(u)_x = \frac{1}{\Delta x}[h(x + \Delta x/2) - h(x - \Delta x/2)].$$

  So do reconstruction on values of $f(u_j)$.
- *Flux splitting*: Required to show stability using Harten.

$$\hat{f}_{j+1/2} = \hat{f}^+_{j+1/2}(u^-) + \hat{f}^-_{j+1/2}(u^+).$$

  Assumptions:
  - $\frac{\mathrm{d}\hat{f}^+}{\mathrm{d}u} \geqslant 0$,
  - $\frac{\mathrm{d}\hat{f}^-}{\mathrm{d}u} \leqslant 0$.

  Lax-Friedrichs is a splittable flux.
- *Limiting/stability*: Focus on $\hat{f}^+$ for now.

$$\hat{f}^{+,\text{mod}}_{j+1/2} = f(u_j) + \text{minmod}(\hat{f}^{+,\text{orig}}_{j+1/2}, \Delta_+ f(u_j), \Delta_- f(u_j))$$

- *Scheme*:

$$u_t = (\hat{f}^+_{j+1/2} + \hat{f}^-_{j+1/2}) - (\hat{f}^+_{j-1/2} + \hat{f}^-_{j-1/2})$$

- *Mesh must be uniform or smoothly mappable to uniform.* WHY?

4 Numerics in Multiple Space Dimensions

- $u_t + f(u)_x + g(u)_y = 0.$

- Weak solutions, entropy solutions same as 1D.
- Motone schemes have the same properties (TVD, entropy condition, L^1 contraction.
- *TVD schemes are at most first order*.
  "Proof": Consider a wiggly jump vs. a straight jump. One has high TV, the other low.
- Saying TVD in nD literature amounts to "TVD in 1D, but straightforwardly generalized to 2D".
- *Maximum principle*: Consider scheme in Harten form. Then $u_{i,j}^{n+1}$ is a convex combination of the values on the stencil.
- *Finite-volume*:

$$\begin{aligned} & \frac{1}{\Delta x \Delta y} \int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} f(u)_x \mathrm{d}x\, \mathrm{d}y \\ = & \frac{1}{\Delta x \Delta y} \int_{y_{j-1/2}}^{y_{j+1/2}} f\left(u(x_{i+1/2}, y, t)\right) - f(u(x_{i-1/2}, y, t))\mathrm{d}y. \end{aligned}$$

  One integral is simple reconstruction, which must be carried out in two directions. Then the second integral must be carried out numerically.
  General procedure:

$$\begin{array}{ccl} & & \{\tilde{u}_{i+1/2,j}\} \xrightarrow{\text{1D rec}} \{u_{i+1/2,j+w_k}\} \longrightarrow \{f(u_{i+1/2,j+w_k})\} \xrightarrow{\text{num.int.}} \{\hat{f}_{i+1/2,j}\} \\ & \overset{\text{1D rec}}{\nearrow} & \\ \{\tilde{\bar{u}}_{i,j}\} & & \\ & \overset{\text{1D rec}}{\searrow} & \\ & & \{\bar{u}_{i,j+1/2}\} \xrightarrow{\text{1D rec}} \{u_{i+\omega_k,j+1/2}\} \longrightarrow \{f(u_{i+\omega_k,j+1/2})\} \xrightarrow{\text{num.int.}} \{\hat{f}_{i,j+1/2}\} \end{array}$$

  Only relevant for third and higher order since

$$\tilde{\bar{u}}_{i,j} = u(x_i, y_j) + O(\Delta x^2, \Delta y^2),$$

  where $\tilde{\cdot}$ and $\bar{\cdot}$ are cell averaging in x and y.
- *Finite-difference*: Generalizes straightforwardly.

5 Systems of Conservation Laws

- *Linear case*:

$$u_t + A\, u_x = 0$$

  A has complete set of eigenvectors and only real eigenvalues, it's called (strongly) hyperbolic.
  If constant linear system, use change of variables and use upwind/downwind depending on sign of eigenvalue. $A^+ = R\Lambda^+ R^{-1} \neq A^{+,\text{elementwise}}$.
- If nonlinear, then find eigenvalues for each new matrix $\nabla \boldsymbol{f}(\boldsymbol{u})$, transform to diagonal form, carry out scalar reconstruction, then transform back.
  Rationale: Separation of shocks–two shocks travelling at different speeds.
- All results about stability and convergence carry over to linear systems using the characteristic procedure above.
- Steps for the nonlinear case:
  - At $x_{j+1/2}$ find a crude "reference vector" $\tilde{u}_{j+1/2}$ as
    - $\tilde{u}_{j+1/2} = \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1})$
    - or *Roe average*: $f(\bar{u}_{j+1}) - f\big(\bar{u}_j\big) = f'(\tilde{u}_{j+1/2})(\bar{u}_{j+1} - \bar{u}_j)$

   - Diagonalize $f'(\tilde{u}_{j+1/2}) = R\,\Lambda R^{-1}$.
   - Transform all involved cell averages using $\bar{v} = R^{-1}\bar{u}$.
   - Carry out 1D reconstruction.
   - Recover $u_{j+1/2} = R\,v_{j+1/2}$.
- For 2D nonlinear system, combine system approach with 2D stuff above.

6 Discontinuous Galerkin

- *Derivation of the Scheme*: Multiply PDE by test function v, integrate by parts, interpret arising boundary terms by comparing with FV, using $v = \mathbf{1}_{I_j}$. This gives

$$\int_{I_j} u_t v - \int_{I_j} f(u) v_x + \hat{f}(u_{j+1/2}^-, u_{j+1/2}^+) - \hat{f}(u_{j-1/2}^-, u_{j+1/2}^+) = 0.$$

  Then pick a basis in the space V_h