EN221 Summary

1 Tensor Stuff

- *Divergence*:

$$\begin{array}{ll}\nabla\cdot\boldsymbol{u}=\partial_i u_i & \int_R \nabla\cdot\boldsymbol{u}=\int_{\partial R}\boldsymbol{u}^T\boldsymbol{n}\mathrm{d}a,\\ \nabla\cdot T=\partial_i T_{ij}\boldsymbol{e}_j & \int_R \nabla\cdot T=\int_{\partial R}T^T\boldsymbol{n}\mathrm{d}a.\\ \nabla\otimes\boldsymbol{u}=\text{Jacobian} & \int_R \nabla\otimes\boldsymbol{u}=\int_{\partial R}\boldsymbol{u}\otimes\boldsymbol{n}\mathrm{d}a,\end{array}$$

  (matrix divergence: columns stay separate)

- *Box product*: $[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]=\boldsymbol{a}\cdot(\boldsymbol{b}\wedge\boldsymbol{c})$
- *Levi-Civita tensor*:

$$\varepsilon_{ijk}=\det\begin{pmatrix}\delta_{i,1}&\delta_{j,1}&\delta_{k,1}\\ \delta_{i,2}&\delta_{j,2}&\delta_{k,2}\\ \delta_{i,3}&\delta_{j,3}&\delta_{k,3}\end{pmatrix}=[\boldsymbol{e}_i,\boldsymbol{e}_j,\boldsymbol{e}_k]=\begin{cases}1 & (ijk)\text{ an even permut. of (123)},\\ -1 & (ijk)\text{ an odd permut. of (123)},\\ 0 & \text{if not.}\end{cases}$$

  $\boldsymbol{e}_j\wedge\boldsymbol{e}_k=\varepsilon_{ijk}\boldsymbol{e}_i$.
  $\det(\boldsymbol{abc})=\varepsilon_{ijk}a_ib_jc_k$

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{ilm}&=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl},\\ \varepsilon_{ijk}\varepsilon_{ijl}&=2\delta_{kl},\\ \varepsilon_{ijk}\varepsilon_{ijk}&=6\end{aligned}$$

- *Principal Invariants*:

$$\begin{aligned}\mathrm{I}_A&=\lambda_1+\lambda_2+\lambda_3=([A\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]+[\boldsymbol{a},A\boldsymbol{b},\boldsymbol{c}]+[\boldsymbol{a},\boldsymbol{b},A\boldsymbol{c}])/[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]=\operatorname{tr}A,\\ \mathrm{II}_A&=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3=([A\boldsymbol{a},A\boldsymbol{b},\boldsymbol{c}]+[A\boldsymbol{a},\boldsymbol{b},A\boldsymbol{c}]+[\boldsymbol{a},A\boldsymbol{b},A\boldsymbol{c}])/[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]=\frac{1}{2}[\operatorname{tr}^2A-\operatorname{tr}A^2],\\ \mathrm{III}_A&=\lambda_1\lambda_2\lambda_3=[A\boldsymbol{a},A\boldsymbol{b},A\boldsymbol{c}]/[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]=\det A.\end{aligned}$$

- *Adjugate/Cofactor of a Tensor*: $A^*(\boldsymbol{a}\wedge\boldsymbol{b})=(A\boldsymbol{a})\wedge(A\boldsymbol{b})\;\Rightarrow\;A^*=\det A(A^{-T})$.
  $\partial_t\det A(t)=\det A\operatorname{tr}((\partial_tA)A^{-1})$
- *Tensor Product*: TO $\otimes$ FROM

$$\begin{aligned}\boldsymbol{e}_i\otimes\boldsymbol{e}_j&=\boldsymbol{e}_i\boldsymbol{e}_j^T\\ (\boldsymbol{u}\otimes\boldsymbol{v})\boldsymbol{a}&=\boldsymbol{u}(\boldsymbol{v}\cdot\boldsymbol{a})\\ (\boldsymbol{u}\otimes\boldsymbol{v})(\boldsymbol{w}\otimes\boldsymbol{x})&=\boldsymbol{v}\cdot\boldsymbol{w}(\boldsymbol{u}\otimes\boldsymbol{x})\\ (\boldsymbol{u}\otimes\boldsymbol{v})A&=\boldsymbol{u}\otimes(A^T\boldsymbol{v})\end{aligned}$$

- *Skewsymmetric matrices*: Rotation around axis $\boldsymbol{\Omega}$ given by orthogonal matrix $Q(t)$.
  $\dot{\boldsymbol{x}}=\dot{Q}\boldsymbol{x}\Rightarrow\partial_t(Q^TQ)=0$,
  $W=\dot{Q}Q^T$, $W=-W^T$. $W\boldsymbol{x}=\boldsymbol{\Omega}\wedge\boldsymbol{x}$.

2 Kinematics

2.1 Static

- *Reference* and *deformed configurations*.

- *Deformation gradient*: assumed regular. $J = \det F \neq 0$.

$$\begin{aligned} \boldsymbol{x}(\boldsymbol{X}) &= \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}), \\ F &= \nabla_{\boldsymbol{X}} \otimes \boldsymbol{x}(\boldsymbol{X}), \\ F^{-1} &= \partial_{x_j} X_\beta \boldsymbol{E}_\beta \otimes \boldsymbol{e}_i \end{aligned}$$

- *Isochoric*: $J = 1$.
- *Polar decomposition*:
  - $F = RU$,
    $F^T F = U^2$, $R = F U^{-1}$.
  - $F = VR$.

  Features:
  - Is unique.
  - R is rotation of principal axes.
  - R average of all rotations.
  - Principal axes of V are $R\boldsymbol{u}_i$.
  - $\sigma(V) = \sigma(U)$.
  - $R = \boldsymbol{v}_k \otimes \boldsymbol{u}_k$.
  - $F = \lambda_k \boldsymbol{v}_k \otimes \boldsymbol{u}_k$.
- *Left/Right Cauchy-Green Deformation Tensor*: $FF^T / F^T F$ SPD.
- *Strain*:

$$\begin{aligned} E &= \frac{1}{2}(F^T F - \mathrm{Id}) \quad (\text{Lagrangean:} \quad |\mathrm{d}\boldsymbol{x}|^2 - |\mathrm{d}\boldsymbol{X}|^2 = 2\mathrm{d}\boldsymbol{X} \cdot E \mathrm{d}\boldsymbol{X}), \\ E' &= \frac{1}{2}(\mathrm{Id} - F^{-T} F^{-1}) \quad (\text{Eulerian:} \quad |\mathrm{d}\boldsymbol{x}|^2 - |\mathrm{d}\boldsymbol{X}|^2 = 2\mathrm{d}\boldsymbol{x} \cdot E' \mathrm{d}\boldsymbol{x}). \end{aligned}$$

- *Stretch*:

$$\lambda(\boldsymbol{M}) = (\boldsymbol{M} \cdot F^T F \boldsymbol{M})^{1/2} = |U\boldsymbol{M}|.$$

  Has local maxima and minima when $\boldsymbol{M}$ is an eigenvector of U.
- *Transformation of area elements*:

$$\boldsymbol{n} \mathrm{d}a = F^* \boldsymbol{N} \mathrm{d}A$$

- *Deformation gradient in cylindrical coordinates*: Given

$$\begin{pmatrix} r \\ \theta \\ z \end{pmatrix} = f(R, \Theta, Z),$$

  we have

$$F = \partial_R \boldsymbol{x} \otimes \boldsymbol{E}_R + \frac{1}{R} \partial_\Theta \boldsymbol{x} \otimes \boldsymbol{E}_\Theta + \partial_Z \boldsymbol{x} \otimes \boldsymbol{E}_Z.$$

  Also expressible as *mixed tensor* from $\boldsymbol{E}_{(R,\Theta,Z)}$ to $\boldsymbol{E}_{(r,\theta,z)}$:

$$F = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R} \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial z} \\ \frac{r}{1} \frac{\partial \theta}{\partial R} & \frac{r}{R} \frac{\partial \theta}{\partial \Theta} & \frac{r}{1} \frac{\partial \theta}{\partial z} \\ \frac{\partial z}{\partial R} & \frac{1}{R} \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{pmatrix}.$$

  Caveat for mixed tensors: $\mathrm{tr}(F) \neq F_{ii}$. However det, V, U as usual. Also works for spherical basis, but more complicated.

2.1.1 Static Examples

- *Pure shear*: $F = \lambda \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \lambda^{-1} \boldsymbol{e}_2 \otimes \boldsymbol{e}_2$.
- *Simple shear*: $F = \mathrm{Id} + \lambda \boldsymbol{e}_1 \otimes \boldsymbol{e}_2$.
- *Pure bending*:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (R-Y)\sin\alpha(x) \\ R-(R-Y)\cos\alpha(x) \\ Z \end{pmatrix}, \quad J = (R-Y)\alpha'.$$
- *Tension and torsion*:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{X}{\sqrt{\lambda}}\cos\left(\frac{\alpha}{l}\lambda Z\right) - \frac{Y}{\sqrt{\lambda}}\sin\left(\frac{\alpha}{l}\lambda Z\right) \\ \frac{X}{\sqrt{\lambda}}\sin\left(\frac{\alpha}{l}\lambda Z\right) + \frac{Y}{\sqrt{\lambda}}\cos\left(\frac{\alpha}{l}\lambda Z\right) \\ \lambda Z \end{pmatrix}.$$
- *Turning a cylinder inside out*.

2.2 Dynamic

- *Steady motion*: $\partial / \partial t\, \boldsymbol{v}(\boldsymbol{x}, t) = 0$.
- *Material/Lagrangean POV*: focus on particle, expressions in terms of $\boldsymbol{X}$ and $t \to$ Solids.
- *Spatial/Eulerian POV*: focus on point in space, expressions in terms of $\boldsymbol{x}$ and $t \to$ Fluids.
- Lines:
  - *Path line*: Curve traced by a fixed particle.
  - *Streamlines*: Field lines of velocity in Eulerian POV.

  Both coincide under steady motion.
- *Material derivative*:
$$\begin{aligned} \dot{\varphi} &= \frac{\partial \varphi}{\partial t} + \nabla_{\boldsymbol{x}} \varphi \cdot \boldsymbol{v}, \\ \dot{\boldsymbol{w}} &= \frac{\partial \boldsymbol{w}}{\partial t} + (\nabla_{\boldsymbol{x}} \otimes \boldsymbol{w}) \boldsymbol{v}, \\ \dot{T} &= \frac{\partial T}{\partial T} + (\nabla_{\boldsymbol{x}} \otimes T) \boldsymbol{v}. \end{aligned}$$
- *Acceleration*: $\boldsymbol{a} = \dot{\boldsymbol{v}}$.
- *Velocity gradient*: $L = \nabla_{\boldsymbol{x}} \otimes \boldsymbol{v} \quad \Rightarrow \quad \dot{F} = L\,F$ (chain rule).
  F requires a "reference state", L does not.
- $\mathrm{d}\dot{\boldsymbol{x}} = \dot{F} \mathrm{d}\boldsymbol{X} = L\,F \mathrm{d}\boldsymbol{X} = L \mathrm{d}\boldsymbol{x}$. Assume $\mathrm{d}\boldsymbol{x} = \boldsymbol{m} |\mathrm{d}\boldsymbol{x}|$.
$$\begin{aligned} \textit{Strain rate:} \quad \frac{|\mathrm{d}\boldsymbol{x}|^{\bullet}}{|\mathrm{d}\boldsymbol{x}|} &= \boldsymbol{m} \cdot L\boldsymbol{m} = \boldsymbol{m} \cdot D\boldsymbol{m} \\ \dot{\boldsymbol{m}} &= L\boldsymbol{m} - \boldsymbol{m}(\boldsymbol{m} \cdot L\boldsymbol{m}) \end{aligned}$$
- *Stretch and Spin*: $L = D + W$, $D = D^T$, $W = -W^T$.
  D_{11}: stretching rate of a line element along the 1-direction
  D_{12}: (roughly) change in angle between the 1- and 2-direction.
  Principal axes $\boldsymbol{p}_i$ of D are rigidly rotating about
$$\boldsymbol{\omega} = \frac{1}{2} \operatorname{curl} \boldsymbol{v}$$
  with $W\boldsymbol{p}_i = \boldsymbol{\omega} \times \boldsymbol{p}_i$.
- *Vorticity*: $\operatorname{curl} \boldsymbol{v} = 2 \cdot$ angular velocity. (Letter here is also $\boldsymbol{\omega}$.)

- $\dot{J} = J\,\mathrm{tr}\,L = J\,\mathrm{div}\,\boldsymbol{v}$.
- *Integrals over moving contours*:

$$\begin{aligned}\oint_{C_t} \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{x} &= \oint_{C_R} \boldsymbol{v}(\boldsymbol{x},t)\cdot F\mathrm{d}\boldsymbol{X}\\ \frac{\mathrm{d}}{\mathrm{d}t}\oint_{C_t} \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{x} &= \oint_{C_R} (\dot{\boldsymbol{v}}(\boldsymbol{x},t)\cdot F + \boldsymbol{v}(\boldsymbol{x},t)\cdot L\,F)\mathrm{d}\boldsymbol{X}\\ &= \oint_{C_t} \dot{\boldsymbol{v}}(\boldsymbol{x},t) + L^T\boldsymbol{v}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\end{aligned}$$

- *Integrals over moving surfaces*: Similar, taking into account that $F^* = J F^{-T}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{S_t} \boldsymbol{u}\cdot\boldsymbol{n}\mathrm{d}s = \int_{S_t} (\dot{\boldsymbol{u}} + \boldsymbol{u}\,\mathrm{tr}(L) - L\boldsymbol{u})\cdot\boldsymbol{n}\mathrm{d}s.$$

- *Integrals over moving volumes/Reynolds' Transport Theorem*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_t} \varphi(\boldsymbol{x})\mathrm{d}v = \int_{R_t} \dot{\varphi} + \varphi\,\mathrm{tr}(L)\mathrm{d}v.$$

  Observe that $\mathrm{tr}(L) = \mathrm{div}\,\boldsymbol{v}$, which is zero in the incompressible case.

- *Circulation*:

$$\begin{aligned}\oint_{C_t} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} &= \int_{S_t} \mathrm{curl}\,\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{s} = \int_{S_t} \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{s}\\ L^T\boldsymbol{v} &= \frac{1}{2}\nabla v^2\\ 0 \overset{\text{if circulation-preserving}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\oint_{C_t} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{x} &= \oint_{C_t} \dot{\boldsymbol{v}}(\boldsymbol{x},t) + L^T\boldsymbol{v}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\\ &= \oint_{C_t} \dot{\boldsymbol{v}}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x} + \oint_{C_t} \frac{1}{2}\nabla v^2\mathrm{d}\boldsymbol{x}\\ &= \oint_{C_t} \boldsymbol{a}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\\ &= \int_{S_t} \mathrm{curl}\,\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{s}\end{aligned}$$

  If $\boldsymbol{a} = \nabla\psi$, then the motion is *circulation-preserving*.
  If circulation-preserving, then

$$\mathrm{curl}\,\boldsymbol{a} = \dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\mathrm{tr}(L) - L\boldsymbol{\omega} = 0.$$

  Then consider the product rule on

$$\frac{\mathrm{d}}{\mathrm{d}t}(J F^{-1}\boldsymbol{\omega}) = \ldots = 0$$

  to find *Cauchy's vorticity formula*:

$$\boldsymbol{\omega} = \frac{1}{J}F\boldsymbol{\omega}_{\mathrm{ref}}.$$

  Field lines of vorticity are *vortex lines*.
  If the motion is circulation-preserving, these are material curves.

3 Balance Laws and Field Equations

- *Conservation of Mass*: Assumption:

$$J\rho = \rho_{\mathrm{ref}} \quad \text{(referential)}.$$

Therefore,

$$\begin{aligned}\dot{\rho}J + \rho\dot{J} &= 0\\ \dot{\rho}J + \rho\, J\,\mathrm{div}\,\boldsymbol{v} &= 0\\ \dot{\rho} + \rho\,\mathrm{div}\,\boldsymbol{v} &= 0\\ \partial_t\rho + \mathrm{div}(\rho\boldsymbol{v}) &= 0.\end{aligned}$$

- *Transport Theorem with density*:
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_t}\rho\Phi\mathrm{d}v = \int_{R_t}\rho\dot{\Phi}\mathrm{d}v.$$
- *Linear Momentum*: $M = \rho\boldsymbol{v}$
  - *Stress vector*: $\boldsymbol{t}_{(\boldsymbol{n})}$ is force/unit area.
  - *Balance law*:
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_t}\rho\boldsymbol{v} = \int_{R_t}\rho\dot{\boldsymbol{v}} = \int_{R_t}\rho\boldsymbol{b}\mathrm{d}v + \int_{\partial R_t}\boldsymbol{t}_{(\boldsymbol{n})}\mathrm{d}a.$$
  - *Stress tensor/Cauchy's Theorem*: $\sigma^T\boldsymbol{n} = \boldsymbol{t}_{(\boldsymbol{n})}$. Derivation:
    - $\boldsymbol{t}_{(-\boldsymbol{n})} = -\boldsymbol{t}_{(\boldsymbol{n})}$ by pillbox and balance law.
    - Tetrahedron argument: $\boldsymbol{n}$ the general normal of the coordinate-system-boxed tetrahedron. Then other faces $a_i = a\boldsymbol{n}_i$, where a is the area of the complicated face. Volume $h\ a/3$. Apply $1/a\cdot$ balance law, let $h\to 0$. Assume continuity, derive linear dependence by assuming values are locally constant.
  - Updated balance law:
$$\int_{R_t}\rho\boldsymbol{a} = \int_{R_t}\rho\boldsymbol{b} + \nabla\cdot\sigma$$
  - *Field equations*:
$$\begin{aligned}\rho_{\mathrm{ref}}\ddot{\boldsymbol{x}} &= \nabla_X\cdot s + \rho_{\mathrm{ref}}\boldsymbol{b} \quad \text{(referential)}\\ \rho\boldsymbol{a} &= \nabla_{\boldsymbol{x}}\cdot\sigma + \rho\boldsymbol{b} \quad \text{(spatial)}\end{aligned}$$
- *Nominal Stress/Conjugate stress*: $s = JF^{-1}\sigma$. ($\sigma^T\boldsymbol{n}\mathrm{d}\boldsymbol{a} = s^T\boldsymbol{N}\mathrm{d}\boldsymbol{A}$ can be directly verified.) Also called *Piola-Kirchoff stress*. s^T is *2nd Piola-Kirchoff stress*.
- *Angular Momentum*: $H = \rho\boldsymbol{x}\wedge\boldsymbol{v}$
  - *Non-polar material*: no contact torques.
  - *Balance law*:
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R_t}\rho\boldsymbol{x}\wedge\boldsymbol{v} \overset{(*)}{=} \int_{R_t}\rho\boldsymbol{x}\wedge\dot{\boldsymbol{v}} = \int_{R_t}\rho(\boldsymbol{x}\wedge\boldsymbol{b} + \boldsymbol{c})\mathrm{d}v + \int_{\partial R_t}\boldsymbol{x}\wedge\boldsymbol{t}_{(\boldsymbol{n})}\mathrm{d}a$$

    $\boldsymbol{c}$ is body torque. Equality $(*)$ follows because $\boldsymbol{x}$-derivatives vanish once $\wedge\boldsymbol{v}$ is applied.
  - Subsituting Cauchy's Theorem into the balance law gives
$$\begin{aligned}\int_{R_t}\boldsymbol{x}\wedge(\nabla_{\boldsymbol{x}}\cdot\sigma + \rho\boldsymbol{b}) &= \int_{R_t}\rho(\boldsymbol{x}\wedge\boldsymbol{b})\mathrm{d}v + \int_{\partial R_t}\boldsymbol{x}\wedge\sigma\boldsymbol{n}\mathrm{d}a\\ \int_{R_t}\boldsymbol{x}\wedge(\nabla_{\boldsymbol{x}}\cdot\sigma) & \qquad \int_{\partial R_t}\boldsymbol{x}\wedge\sigma\boldsymbol{n}\mathrm{d}a\end{aligned}$$

    View in component form, apply Gauß, derive $\varepsilon_{ijk}\sigma_{ji} = 0 \Rightarrow \sigma = \sigma^T$.
  - *Field equations*:
$$\begin{aligned}s^TF^T &= Fs \quad \text{(referential)}\\ \sigma^T &= \sigma \quad \text{(spatial)}\end{aligned}$$

- *Vector identities*:

$$\begin{aligned}(\boldsymbol{A}\cdot\nabla)\boldsymbol{A} &= \frac{1}{2}\nabla|\boldsymbol{A}|^2+(\nabla\times\boldsymbol{A})\times\boldsymbol{A}\\ (\boldsymbol{A}\cdot\nabla)\boldsymbol{A} &= (\nabla\otimes\boldsymbol{A})\boldsymbol{A}.\end{aligned}$$

  Use these identities to rewrite the $\dot{\boldsymbol{v}}$ as $\nabla(\boldsymbol{v}^2)$ for irrotational flow.

- *Types of fluid flow*:
  - *Inviscid*: $\sigma=-p\,\mathrm{Id}\Rightarrow\operatorname{div}\sigma=-\nabla p$.
  - *Incompressible*: $\dot{\rho}=0$ or $\operatorname{div}\boldsymbol{v}=0$.
  - *Steady*: $\partial_t\boldsymbol{v}=\boldsymbol{0}$.
    $\dot{\rho}=\boldsymbol{v}\cdot\nabla\rho$.
  - *Irrotational*: $\boldsymbol{\omega}=\boldsymbol{0}$ or $\boldsymbol{v}=\nabla\varphi$.
  - *Elastic*: $p(\rho)$
  - *Ideal=incompressible*: $\operatorname{div}\boldsymbol{v}=0$, $J=1$.

- *Rayleigh-Plesset equation*: Begin with deformation of spherical shell (with extent!), assume $J\equiv 1$. Derive ODE.

- *Conservative potentials*: $\boldsymbol{b}=-\nabla\beta$
  - Elastic or ideal flow here is circulation preserving, i.e. $\boldsymbol{a}=-\nabla$something.
    - Have

$$\begin{aligned}\boldsymbol{a} &= -\frac{1}{\rho}\nabla p(\rho)+\boldsymbol{b}\\ &= -\frac{1}{\rho}p'(\rho)\nabla\rho-\nabla\beta.\end{aligned}$$

    - Define

$$\begin{aligned}\varepsilon(\rho) &= \int_0^\rho\frac{1}{\rho'}p'(\rho')\mathrm{d}\rho'\\ \Rightarrow\nabla\varepsilon(\rho) &= \varepsilon'(\rho)\nabla\rho\end{aligned}$$

    - Therefore

$$\boldsymbol{a}=-\nabla(\varepsilon(\rho)+\beta).$$

    - For ideal fluid substitute p/ρ_0 for ε.
  - *Bernoulli's Theorem*:
    - Flow irrotational (i.e. $\boldsymbol{v}=-\nabla\varphi$):

$$\nabla\left(\partial_t\varphi+\frac{v^2}{2}+\varepsilon(\rho)+\beta\right)=\boldsymbol{0}.$$

      Proof: Just rewrite, obtaining $v^2/2$ from second term of material derivative.
    - Flow steady:

$$\left(\frac{v^2}{2}+\varepsilon(\rho)+\beta\right)^{\bullet}=0,$$

      i.e. this quantity is constant along streamlines.
      Proof: Exploit $\boldsymbol{v}\cdot\dot{\boldsymbol{v}}=\boldsymbol{v}\cdot\nabla(\boldsymbol{v}^2)$
    - Flow both irrotational and steady:

$$\nabla\left(\frac{v^2}{2}+\varepsilon(\rho)+\beta\right)=\boldsymbol{0}.$$

- *Acoustic wave equation*:
  - Assume $\rho = \rho_0 + \delta\rho$, $|\boldsymbol{v}| \ll 1$, $|\nabla \boldsymbol{v}| \ll 1$.
  - Start with $\partial_t (\nabla \cdot \boldsymbol{v})$, use cons. of. momentum without second order term, cons. of mass as $\partial_t \rho + \rho_0 \mathrm{div}(\boldsymbol{v}) = 0$.
  - $\delta\rho_{tt} = c^2 \nabla^2 \delta\rho$, with $c = \sqrt{\partial_\rho p}$.
- *Mach number*: assume steadiness $\boldsymbol{b} = 0$, use $\boldsymbol{v} \cdot \dot{\boldsymbol{v}}$ in terms of c^2.

$$\boldsymbol{v} \cdot (\rho \boldsymbol{v})^\bullet = \boldsymbol{v} \cdot \dot{\boldsymbol{v}} \rho \left(1 - \underbrace{\frac{\boldsymbol{v}^2}{c^2}}_{m:=} \right)$$

  - Supersonic nozzle $m < 1$, $m > 1$.
- *Conservation of Energy*:
  - *Balance law*:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} K(R_t) &= -S(R_t) + P(R_t) \\ \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\frac{1}{2} \int \rho \boldsymbol{v} \cdot \boldsymbol{v} \mathrm{d}v}_{\text{Kinetic energy } K(t)} &= - \underbrace{\int_{R_t} \mathrm{tr}(\sigma D) \mathrm{d}v}_{\text{Stress power } S(R_t)} + \underbrace{\int_{R_t} \rho \boldsymbol{b} \cdot \boldsymbol{v} + \int_{\partial R_t} \boldsymbol{\sigma n} \cdot \boldsymbol{v} \mathrm{d}a}_{\text{Power supplied } P(R_t)} \end{aligned}$$

    Proof: Multiply Equation of Motion by $\boldsymbol{v}$, integrate by parts in the σ term.
  - *Field equation*:

$$\underbrace{\rho \left(\frac{1}{2} v \cdot v \right)^\bullet}_{\text{Kinetic Energy}} + \underbrace{\mathrm{tr}(\sigma D)}_{\text{Stress Power}} = \underbrace{\nabla_{\boldsymbol{x}} \cdot (\sigma \boldsymbol{v}) + \rho \boldsymbol{b} \cdot \boldsymbol{v}}_{\text{Rate-of-working}}.$$

  - Key words for more global energy conservation: internal energy $U(R_t)$, heat supply per unit mass $H(R_t)$, heat flux through material surface.

$$\frac{\mathrm{d}}{\mathrm{d}t} \{K + U\} = P(R_t) + H(R_t).$$

    Now, because there is a stress power loss above, there needs to be a gain here:

$$\frac{\mathrm{d}}{\mathrm{d}t} U(R_t) = S(R_t) + H(R_t).$$

- *Jump conditions*: For the balance law

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{R_t} \rho \pi = \int_{R_t} \rho s + \int_{\partial R_t} f_{(\boldsymbol{n})},$$

we get

$$[\rho V \pi + f_{(\boldsymbol{n})}] = 0.$$

V_n interface speed, $V = V_n - \boldsymbol{v} \cdot \boldsymbol{n}$.

		Mass	Mom.	A.Mom.	Energy
π	quantity per unit mass	1	$\boldsymbol{v}$	$\boldsymbol{x} \wedge \boldsymbol{v}$	$\varepsilon + \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v}$
s	supply of π per unit mass	0	$\boldsymbol{b}$	$\boldsymbol{x} \wedge \boldsymbol{b}$	$\boldsymbol{b} \cdot \boldsymbol{v} + r$
$f_{(\boldsymbol{n})}$	influx of π per unit area	0	$\boldsymbol{t}_{(\boldsymbol{n})}$	$\boldsymbol{x} \wedge \boldsymbol{t}_{(\boldsymbol{n})}$	$\boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v} + h_{(\boldsymbol{n})}$

so that for example

$$\begin{aligned} [\rho V] &= 0, \\ [\rho V \boldsymbol{v} + \boldsymbol{t}_{(\boldsymbol{n})}] &= \boldsymbol{0}. \end{aligned}$$

Or for material jumps: $[\boldsymbol{t}_{(\boldsymbol{n})}] = \boldsymbol{0}$.

Derivation:

- Modification for moving boundary is

$$-\int_{\text{jump surface}} [\rho\pi] V_n.$$

- Then use pillbox that flattens around surface.

Examples:

- Free boundary: pressure must be continuous, because otherwise there's a finite force on something massless.

- *Stokes waves*:
  - Assume $\boldsymbol{v} = \nabla\varphi$.
  - Conservation of mass $\nabla^2\varphi = 0$.
  - Bernoulli's equation

$$\partial_t\varphi + \frac{v^2}{2} + \frac{p}{\rho_0} + \beta = \text{const}$$

  - BCs: z depthward, $z = \eta$ free surface
    - $\varphi_z = 0$ at bottom
    - $\frac{\mathrm{d}}{\mathrm{d}t}(z - \eta) = 0$ at $z = \eta \to \partial_t\varphi = \partial_t\eta$ at $z = 0$ (!).
    - pressure continuous at interface. Use Bernoulli's equation to rewrite as condition

$$\partial_t\varphi + g\eta = 0 \quad \text{at } z = 0.$$

  - *Surface tension*: $p_1 - p_2 = -\gamma \cdot \text{curvature}$.
  - *Rayleigh-Taylor instability*: Large density over small density.
  - *Kelvin-Helmholtz instability*: Wave formation.

4 Constitutive Laws

- *Observer*: A reference frame/coordinate system w.r.t. which vectors and tensors are seen.

$$\boldsymbol{x}^* = \boldsymbol{c}(t) + Q(t)\boldsymbol{x}$$

  so, for example, $F^* = Q\,F$, $J^* = J$, $U^* = U$, $R^* = Q\,R$.

- *Objective fields*:

$$\begin{array}{rcl} \varphi^*(\boldsymbol{x}^*) & = & \varphi(\boldsymbol{x}) \\ \boldsymbol{u}^*(\boldsymbol{x}^*) & = & Q\boldsymbol{u}(\boldsymbol{x}) \\ A(\boldsymbol{x}^*) & = & Q\,A(\boldsymbol{x})Q^T \end{array}$$

  Examples: D, regions, normals, σ
  Non-examples: $\boldsymbol{v} = \dot{\boldsymbol{c}} + Q\,\boldsymbol{v}$, $L = Q\,L\,Q^T + \dot{Q}Q^T$, W.

- *Constraint stress*:
  - Must be workless, i.e. $\operatorname{tr}(N\,D) = 0$
  - Constraint given as $\lambda(C) = 0 \to \dot{\lambda} = \operatorname{tr}(\lambda_C\dot{C}) = 0$, where $C = F^T F$.
  - $\dot{C} = 2F^T D\,F \Rightarrow N = \alpha F\lambda_C F^T$.

- *Fluid*: $\sigma = g(L)$.

Cannot support shear stress at equilibrium. If *ideal*, also cannot support shear stress when in motion.

- *Objectivity*: $\sigma^* = g(L^*)$.
  - Choose $Q = \mathrm{Id}$, $\dot{Q} = -W$ to obtain that $g(L) = g(D)$.
- Most general such g:

$$\sigma(D) = \alpha I + \beta D + \gamma D^2,$$

  with α, β, γ functions of the invariants of D.
  Proof: Cayley-Hamilton.
- *Incompressible fluid*:

$$\sigma = -p\,\mathrm{Id}.$$

- *Ideal fluid*:

$$\sigma = -p(\rho)\mathrm{Id}$$

- *Newtonian fluid*:

$$\sigma = -p(\rho)\mathrm{Id} + 2\mu D$$

- *Navier-Stokes equation*:

$$\rho \boldsymbol{a} = -\nabla p + \mu \Delta \boldsymbol{v} + \rho \boldsymbol{b}$$

  plus conservation of mass.
- *Rescaling* $\tilde{x} = x/l$, $\tilde{\boldsymbol{v}} = \boldsymbol{v}/v$, $p = p/(\rho_0 v^2)$, $\tilde{t} = t/l$.
  Then *kinematic viscosity* is $\nu = \mu/p$.
- *Reynolds number*: $\mathrm{Re} = l\,v/\nu$.
  High: Dominated by inertial effects.
  Low: Dominated by viscous effects.
- No-slip BCs apply only for viscous fluids.
- *Wiggling plate*: Watch for emergence of a boundary layer.

• *Solid*: $\sigma = f(F)$

- *Material Symmetry*: $P \in \mathcal{S}$, where $\mathcal{S}$ is the symmetry group of the material.

$$\sigma = f(F) = f(FP)$$

  Isotropic Material: $\mathcal{S} = \mathrm{SO}(3)$. Then choose $P = R^T \Rightarrow \sigma = f(F) = f(V)$.
- *Objectivity*: $\sigma^* = f(V^*)$.
  Most general expression to satisfy this:

$$\sigma(V) = \alpha \mathrm{Id} + \beta V + \gamma V^2,$$

  with α, β, γ functions of the invariants.
- *Lamé constant/Young's Modulus*: Linearization!

$$\begin{aligned} F &= \mathrm{Id} + \nabla \boldsymbol{u} \\ E &= \frac{1}{2}\big[F^T F - \mathrm{Id}\big] \approx \frac{1}{2}\big[\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T\big] \\ V &\approx \mathrm{Id} + E \\ R &\approx \mathrm{Id} + \frac{1}{2}\big[\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^T\big] \end{aligned}$$

  Use these in

$$\sigma = c_0 \mathrm{tr}\, V\, \mathrm{Id} + c_1 V + c_3 V^2 \approx \lambda \mathrm{tr}(E)\mathrm{Id} + 2\mu E,$$

  where λ, μ are the *Lamé constants*.

- *Strain energy per volume*: $W(F) \leftarrow$ the usual way to specify constitutive relations for solids
  Then
  $$\sigma = \frac{1}{J} \cdot \underbrace{\frac{\partial W}{\partial F}}_{\frac{\partial W}{\partial V} R} F^T = \frac{1}{J} \cdot \frac{\partial W}{\partial V} V$$
  Invoke objectivity: $W(F) = W(U)$
  Invoke isotropy: $W(F) = W(V)$
  $\Rightarrow W$ depends only on invariants of V.
  $\Rightarrow$ W's principal axes line up with those of V, i.e. *principal stresses* $\|$ *principal stretches*:
  $$\sigma_\alpha = \frac{1}{J} \lambda_\alpha \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_\alpha}.$$
  Incompressible:
  $$\sigma_\alpha = \frac{1}{J} \lambda_\alpha \frac{\partial W(\lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_\alpha} - p.$$
  Specifying W in terms of $B = F F^T$:
  $$\sigma = \frac{2}{J} \big(\mathrm{III}_B W_{\mathrm{III}_B} \, \mathrm{Id} + (W_{\mathrm{I}_B} + \mathrm{I}_B W_{\mathrm{II}_B}) B - W_{\mathrm{II}_B} B^2 \big),$$
  where subscripts by $\mathrm{I}_B, \mathrm{II}_B, \mathrm{III}_B$ mean partial derivatives.

- *Neo-Hookean material*:
  $$\begin{aligned} W &= \frac{1}{2} \mu \big[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2 \ln(J) \big] + \frac{1}{2} \mu' (J - 1)^2, \\ \text{unconstrained:} \quad \sigma_i &= \mu(\lambda_i^2 - 1) + \mu' J \, (J - 1), \\ \text{incompressible:} \quad \sigma_i &= \mu \lambda_i^2 - p. \end{aligned}$$

- Solving a solids problem:
  - Calculate F (Kinematics)
  - Calculate $B = F F^T$
  - Calculate σ
  - Apply conservation of momentum in deformed configuration. Solve for unknowns $\boldsymbol{x}$, p, using BCs.