PDE Summary

1 General Stuff

- *Standard mollifier*:
$$\eta(x) = \exp\left(\frac{1}{x^2-1}\right)\mathbf{1}_{[-1,1]}$$
is a C_c^∞ hump.
$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(x/\varepsilon).$$
Normalization ($\int \ = 1$) is still missing.
- *Gamma function*:
$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \mathrm{d}t.$$
- *Volumes of sphere and ball*:
$$\begin{aligned} |S^{n-1}| &= \omega_n r^{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}. \\ |B^n| &= \frac{\omega_n}{n} r^n. \end{aligned}$$
- *Green's identities*:
$$\begin{aligned} \int_U v\Delta u &= -\int_U \nabla v \cdot \nabla u + \int_{\partial U} v\partial_n u \\ \int_U v\Delta u - u\Delta v &= \int_{\partial U} v\partial_n u - u\partial_n v \end{aligned}$$
- *Young's Inequality*:
$$\|f * g\|_{L^r} \leqslant \|f\|_{L^p} \|g\|_{L^q} \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$
In particular $q = 1$, $r = p$.
- *Generalized Hölder*:
$$\|f_1 \cdot f_2 \cdots f_m\|_{L^1} \leqslant \|f_1\|_{p_1} \|f\|_{p_2} \cdots \|f_m\|_{p_m}$$
if
$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.$$
- *Interpolation Inequality for L^p*: If $1 \leqslant s \leqslant r \leqslant t \leqslant \infty$
$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t},$$
$u \in L^s \cap L^t$, then $u \in L^r$ and
$$\|u\|_{L^r} \leqslant \|u\|_{L^s}^\theta + \|u\|_{L^t}^{1-\theta}.$$
- *Compact*: Every open cover has finite subcover. Metric space: $\Leftrightarrow$ sequentially compact. Heine-Borel (finite-dim): $\Leftrightarrow$ closed and bounded.
- *Arzelà-Ascoli*: (S, d) compact metric space. $M \subset C(S)$ with sup-norm is compact if M is bounded, closed and equicontinuous.
- *Precompact*: has compact closure.
- *Compact operator*: $T: B_1 \to B_2$ *compact* if T continuous and $T(A)$ precompact for every bounded A.
- *Fredholm Alternative*: $T: B \to B$ linear, continuous, compact:
  - either $(I - T)x = 0$ has a nontrivial solution

  - or $(I-T)^{-1}$ exists and is bounded.

  "Uniqueness and Compactness ⇒ Existence".

- *Lax-Milgram*: $B: H \times H \to \mathbb{F}$, bounded above and coercive ⇒ $B[u, g] = F(u)$ solvable in H for every $g \in H$.
  Proof: Build operator $T_g: H \to H^*$ that gives $T_g(u) = B[u, g]$ (Riesz rep.). Prove 1-1 and onto.
  The point is: *no symmetry*.

- *Banach-Steinhaus/Uniform Boundedenss Principle*:
  X BR, Y NR, $T_i \in L(X, Y)$ $(i \in I)$, $\sup_{i \in I} \|T_i x\| < \infty$ $(x \in X)$
  $\Rightarrow \sup_i \|T_i\| < \infty$.
  Read as "linear+pw bounded ⇒ uniformly bounded."

2 Equations

- *Classification of second order equations*:

$$A_{i,j}\partial_i\partial_j u + B_i\partial_i u + C = 0,$$

  where A is symmetric WLOG can be rewritten into one of

$$\begin{aligned} u_{xx} + u_{yy} + \text{l.o.d.} &= F, \\ u_{xx} - u_{yy} + \text{l.o.d.} &= F, \\ u_{xx} \pm u_y + \text{l.o.d.} &= F. \end{aligned}$$

- *Minimal surface equation*:

$$\operatorname{div}\left(\frac{Du}{\sqrt{|Du|^2+1}}\right) = 0$$

- *Monge-Ampére equation*:

$$\det(D^2u) = K(x)(1+|Du|^2)^{(n+2)/2}$$

3 Laplace's Equation

U open.

- $u \in C^2(U)$: harmonic, subharmonic $\Delta u \geqslant 0$, superharmonic.
- *Mean Value Inequality:* u subharmonic

$$\begin{aligned} u(x) &\leqslant \fint_{S(x,r)} u(y)\mathrm{d}S_y \\ u(x) &\leqslant \fint_{S(x,r)} u(y)\mathrm{d}B \end{aligned}$$

  (implies *Mean Value Property* if harmonic)
  Proof: $0 \leqslant \int_B \Delta u = \int_S \partial_n u$, then exploit $\partial_n u = \partial_r(x + \rho n)$. $\int_B u = \int_r \int_{|\omega|=1} u = u \int_r$.

- *Strong Maximum Principle:* U bounded, connected, u subharmonic, $u(x) = \sup_U u \Rightarrow u$ constant
  Proof: Consider $\{u = \sup\}$. By MVI, $u = \sup$ on any ball in U. Thus $\{u = \sup\}$ open. But so is $\{u < \sup\}$. $U = \{u = \sup\} \cup \{u < \sup\}$, both open ⇒ $\{u = \sup\} = U$.

- *Weak maximum principle:* $u \in C(\bar{U})$ and subharmonic. Then u assumes extrema on the boundary.
  Proof: SMP or: Suppose $x \in U$ is max and $\Delta u > 0$. Then $Du = 0$ and D^2u negative semidef, contradicting $\Delta u = \operatorname{tr}(D^2u) \geqslant 0$. If only $\Delta u \geqslant 0$, consider $u + \varepsilon|x|^2$, which is strictly subharmonic.

- Strong ⇒ constant, Weak ⇒ extrema on boundary.

- Uniqueness follows directly from the WMP.
- *Harnack's Inequality:* $u \geqslant 0$ (!) harmonic, $U' \subset\subset U$ connected $\Rightarrow \exists C$ such that $\sup u < C \inf u$.
  Proof: Pick x_1, $x_2 \in U$, apply MVP for large and small circle, respectively, then shrink/expand domain by using $u \geqslant 0$, take sup/inf. Use cover of balls to repeat argument as necessary.
- *Fundamental solution:* look for radial symmetry

$$\psi = C + \begin{cases} \frac{1}{2\pi} \log r & n = 2, \\ \frac{1}{(2-n)\omega_n} r^{2-n} & n \geqslant 3. \end{cases}$$

  Constant chosen because it gives the right constant to prove $\Delta\psi = \delta_0$ (use Green's second id on a ball surrounding the signularity). $K(x, \xi) = \psi(|x - \xi|)$.
- *Liouville's Theorem:* (only in 2D) Subharmonic functions bounded above are constant.
- $u \in C^2(\bar{U})$:

$$u(\xi) = \int_U K(x, \xi) \Delta u \mathrm{d}x + \int_{\partial U} u \partial_{n_x} K(x, \xi) - K(x, \xi) \partial_{n_x} u \mathrm{d}S_x. \tag{1}$$

  Proof: Integrate on $U \setminus B_\varepsilon$, $\varepsilon \to 0$.
  Remains valid if K replaced by $K + w$ with harmonic w.
- *Green's function for Dirichlet problem:* $\Delta_x G = \delta_\xi$, $G(x, \xi) = 0$ for $x \in \partial U$. Use G in (1). To get one, we need to find w with $w = -K$ on ∂U. (Use method of images.) For a ball, we get the *Poisson kernel*

$$H(x, \xi) = \frac{r^2 - |\xi|^2}{\omega_n r |x - \xi|^n}$$

  Poisson's integral formula:

$$u(\xi) = \int_{S(0,r)} H(x, \xi) f(x) \mathrm{d}S_\xi.$$

- *Kelvin's transformation*: u harmonic $\Rightarrow$

$$|x|^{2-n} u(x/|x|^2) \text{ harmonic for } x \neq 0.$$

- Properties of H:
  - $H(x, \xi) = H(\xi, x)$
  - $H(x, \xi) > 0$ on $B(0, r)$
  - $\Delta_\xi H(x, \xi) = 0$ for $\xi \in B(0, r)$ and $x \in S(0, r)$
  - $\int_{S(0,1)} H(x, \xi) \mathrm{d}S_x = 1$
- *Existence on a ball:* also gives $C(\bar{B})$
  Proof: Differentiate under integral (using DCT). Prove continuity onto the boundary by

$$u(\xi) - f(y) = \int_{S(\xi,r)} H(x, \xi)(f(x) - f(y)) \mathrm{d}S_x$$

  Use ε-δ-continuity of f and split integral into $|x - y| < \delta$ and $|x - y| > \delta$. (Method called *approximate identities*.)
- *Converse of MVP*: $u \in C(U)$ harmonic $\Leftrightarrow$ satisifes MVP for every $B(x, r) \subset U$.
  Proof: Construct a harmonic function v on $B(x, r)$ with $v = u$ on $S(x, r)$. $v - u$ satisfies MVP on any subcircle, thus it satisfies the strong maximum principle. Thus $v = u$.
- *Real analytic:* completely represented by absolutely convergent Taylor series.
  $\exists M > 0 \forall \alpha$: $|\partial^\alpha f(y)| \leqslant \frac{M|\alpha|!}{r^{|\alpha|}} \Leftrightarrow$ analytic.
  Real analytic f is completely determined by power series (use similar open-set method on $\{\partial^\alpha h(y) = 0 \forall \alpha\}$ as SMP)

- *Harmonic ⇒ Analytic:* Consider $H(x, \xi + i\sigma)$. Find a region of σ where H is differentiable.
- Analyticity estimates can be obtained by the MVP applied to $\partial_{x_j} u$, then coordinatewise Gauß, giving

$$|\partial_{x_j} u(x)| \leqslant \frac{n}{r} \max_{S(x,r)} |u| \leqslant \frac{n}{r} \sup_U |u|.$$

  Then iterate this estimate with $1/|\alpha|$ radius to get

$$|\partial^\alpha u(x)| \leqslant \left(\frac{n|\alpha|}{r}\right)^{|\alpha|} \max_{S(x,r)} |u|.$$

- Uniformly (on compact subsets of U) converging sequences of harmonic functions converge to harmonic functions.
  Proof: Limit is continuous (because of uniform convergence). Now exchange limits (DCT) in MVP and prove harmonicity.
- *Harnack's convergence theorem:* u_k harmonic, increasing and bounded at a point. Then (u_k) converges uniformly on compact subsets to a harmonic function.
  Proof: above + Harnack inequality.
- *"Montel's Theorem"*–a compactness criterion:
  (u_k) bounded, harmonic ⇒ ∃uniformly (on compact subsets) converging subsequence → harmonic limit.
  Proof: (u_k) is equicontinuous because of the derivative estimates and the assumed uniform bound.
- *Subharmonicity on C(U):* Satisfies MVI locally.
- Perron's method:
  - $S_f := \{v \in C(\bar{U}), v \leqslant \mathrm{BC}, v \text{ subharmonic}\}$.
  - $u := \sup S_f$ is harmonic.
    Proof:
    - S_f is closed under finite max. (MVI)
    - *Harmonic lifting:* v subharmonic,

$$V(x) = \begin{cases} \text{harmonic function with matching BCs} & B(\xi, r), \\ v & \text{elsewhere.} \end{cases}$$

      $v \in S_f \Rightarrow V \in S_f$, $v \leqslant V$.
    - Fix a closed ball, grab sequence $v_k \to u$ at a point ξ. $\bar{v}_k := \max(v_1, \ldots, v_k, \min \mathrm{BC})$.
    - Replace these by their harmonic lifting V_k around ξ.
    - HCT for a limit V.
    - Prove $V = u$ on ball by finding SMP uniqueness of harmonic liftings of in-between $(V < u)$ functions.
- *Barrier function at $y \in \partial U$/regular boundary point:*
  $w \in C(\bar{U})$ subharmonic, $w(y) = 0$, $w(\partial U \setminus \{y\}) < 0$.
  ∃tangent plane ⇒ regular
  ∃exterior sphere ⇒ $\text{barrier} = K(\text{boundary point}, \text{outside center}) - K(x, \text{outside center})$
  ∃exterior cone ⇒ regular
- At regular boundary points, $u = \mathrm{BC}$.
  Proof:
  - Fix $\varepsilon > 0$. δ from ε-δ with f.
  - $v = \mathrm{BC} + A \cdot \text{barrier} - \varepsilon$, where $A\, w \leqslant -2 \max \mathrm{BC}$ outside a ball around the boundary point in question. v subharmonic by def.

    - Show $v \leqslant f(x)$ on boundary and interior.
    - Do some funky tricks involving $-f$, its Perron function, and the maximum principle to show opposite inequality.

- The Dirichlet problem is solvable for all continuous BC data iff the domain is regular.

3.1 Energy Methods

- $0 = \int w \Delta w = \int |\nabla w|^2$ proves uniqueness in $C^2(\bar{U})$.
- *Energy Functional:*
$$I[w] = \int_U \frac{1}{2} |\nabla w|^2 + w\, g \mathrm{d}x$$
for g the RHS.
- *Dirichlet's principle:* $u \in C^2(\bar{U})$ solves PDE+BC $\Leftrightarrow$ it minimizes $I[u]$ over $\{w \in C^2(\bar{U}), w = \text{RHS on } \partial\Omega\}$.
Proof: PDE $\Rightarrow$ min: Start from
$$0 = \int (-\Delta u + g)(u - w),$$
use Gauß, Cauchy-Schwarz, $\sqrt{a}\sqrt{b} \leqslant 1/2(a^2 + b^2)$.
min $\Rightarrow$ PDE: $w = u + t\, v$, for $v \in C_c^\infty$. Differentiate by t.

3.2 Potentials

- *Potential of a measure*:
$$u_\mu(x) = \frac{2-n}{\omega_n} \int_{\mathbb{R}^n} K(x, y) \mu(\mathrm{d}y) = \int_{\mathbb{R}^n} |x - y|^{2-n} \mu(\mathrm{d}y)$$
- Computable for a sphere with uniform charge density (same as point charge), finite line, disk.
- $u_\mu = 0 \Rightarrow \mu = 0$.
Proof: Show $\mu * f = 0$ for any $f \in C_c^\infty$ by
$$\mu * f = \mu * (K * \Delta f) = (\mu * K) * \Delta f = 0.$$
- *Potentials of compact set*: Harmonic function with BC 1 on compact set F and BC zero at infinity. Perron function on ever-increasing balls–independent of exact domains.
- *A (unique) generating (positive) measure on ∂F exists:*
Proof (if $\partial F \in C^2$): by Poisson's boundary representation formula (with both u and $\partial_n u$)
$$p_F(\xi) = \int_{\partial F} K(x, \xi) \underbrace{\partial_n p_F \mathrm{d}S_x}_{\text{measure!}}.$$
$\partial_n u \leqslant 0$ by the max principle (1 on the boundary must be the max value) $\Rightarrow$ positivity.
Proof (if not):
    - Approximate F through shrinking compact sets with C^∞ boundary ($1/k^2$-mollified indicators of $F^{1/k} = \{\mathrm{dist}(x, F) \leqslant 1/k\}$. $\psi = \varphi_{1/k^2} * \mathbf{1}_{F^{1/k}}$. Then consider $F^{1/2k} \subset \psi^{-1}([c, 1]) \subset F^{1/k}$ and use Sard's Theorem to deduce boundary smoothness for a.e. c. Generate μ_k by above theorem.
    - $p_{F_k} \to p_F$ uniformly on compact subsets (Harnack)
    - Prove $\mu_k(\mathbb{R}^n) \leqslant R^{n-2}$ by using a $B(0, R) \supset F_k$–use Fubini and the generator of the disk potential. ("*Gauß' trick*') Thus $\exists$weak-* convergent subsequence supported on ∂F. Thus convergene of $p_{F_k} \to p_F$ away from ∂F. Uniqueness by uniqueness of potentials of measures.

3.3 Lebesgue's Thorn

- In 2D, Riemann mapping theorem guarantees that point regularity is topological, not geometric.
- Lebesgue's Thorn: Using level sets of the potential of the measure $x^\beta \mathrm{d}x$ on (0, 1), one may construct exceptional points.

3.4 Capacity

- $$\mathrm{cap}(F) = \mu_F(\mathbb{R}^n) = \frac{2-n}{\omega_n} \int_{\partial F \text{ or enclosing surface}} \partial_n p_F \mathrm{d}S_x.$$
- If $\partial F \in C^2$, Green's 1st id gives
$$\mathrm{cap}(F) = \frac{2-n}{\omega_n} \int_{U \subset \mathbb{R}^n \setminus F} |\nabla p_F|^2.$$
- *Wiener's criterion:* $y \in \partial U$ regular $\Leftrightarrow$
$$\lambda^{2-n} \sum_{k=0}^{\infty} \lambda^{k(2-n)} \mathrm{cap}(F_k) \qquad F_k := \{\lambda^{k+1} \leqslant |x-y| \leqslant \lambda^k\} \qquad (\lambda \in (0,1)).$$
- Properties of capacity:
  - $F_1 \subset F_2 \Rightarrow \mathrm{cap}(F_1) \leqslant \mathrm{cap}(F_2)$ (*Gauß' Trick!*)
  $$\mathrm{cap}(F_1) = \int_{\mathbb{R}^n} \mu_1(\mathrm{d}x) = \int_{\mathbb{R}^n} p_2 \mu_1(\mathrm{d}x) = \int\int |x-y|^{2-n} \mu_2(\mathrm{d}y) \mu_1(\mathrm{d}y) = \int p_1 \mu_2(\mathrm{d}y) \leqslant \mathrm{cap}(F_2).$$
  - (F_k) nested sequence with $\bigcap F_k = F$, then $\mathrm{cap}(F_k) \to \mathrm{cap}(F)$.
  (smooth $\varphi = 1$ on F_1, $\mathrm{cap}(F) = \int \varphi \mu_F \leftarrow \int \varphi \mu_{F_k} = \mathrm{cap}(F_k)$)
  - $\mathrm{cap}(A \cup B) \leqslant \mathrm{cap}(A) + \mathrm{cap}(B)$.
  ($p_\cup \leqslant p_A + p_B$ by WMP. Then use Gauß' trick.)
  - $\mathrm{cap}(A \cup B) + \mathrm{cap}(A \cap B) \leqslant \mathrm{cap}(A) + \mathrm{cap}(B)$
- $\mathrm{cap}(\overline{B(0,R)}) = \mathrm{cap}(S(0,R)) = R^{n-2}$.
- *Screening*: nested spheres $A \subset B$. $\mathrm{cap}(A \cup B) = \mathrm{cap}(B)$ (think of the potentials)
- $\mathrm{cap}(F) = \sup \{\mu(F) : \mathrm{supp}(\mu) \subset F, u_\mu(F) \leqslant 1\}$ (Smooth approx F_k to F so that $p_{F_k} = 1$ on ∂F. Then Gauß' trick.)
- *Coulomb energy*:
$$E[\mu] = \frac{1}{2} \int\int |x-y|^{2-n} \mu(\mathrm{d}x) \mu(\mathrm{d}y).$$
Mutual energy:
$$E[\mu, \nu] = \frac{1}{2} \int\int |x-y|^{2-n} \mu(\mathrm{d}x) \nu(\mathrm{d}y).$$
- Properties:
  - If $E[|\mu|] < \infty$, then pos.def.
  - CSU
  - $\mu \mapsto E[\mu]$ strictly convex
- *Gauß' principle:* $\mu \geqslant 0$ finite measure on F.
$$G[\mu] = E[\mu] - \mu(F) \geqslant -\frac{1}{2} \mathrm{cap}(F)$$

Proof:

- $G(\mu)$ bounded below (F compact $\Rightarrow |x-y|$ bdd.)
- Infimizing sequences are precompact (i.e. have bounded $\mu_k(F)$)
- G is wlsc (take infimizing sequence (μ_k), use max $(M, |x-y|)$ to cut off, $k\to\infty$, $M\to\infty$ (MCT), consider $E[\mu-\mu_k]$)
- Minimizer is unique (strict convexity)
- Minimizer is μ_F (Consider Euler-Lagrange Equation)
- Evaluate minimum

- *Kelvin's principle:*

$$\frac{1}{2\mathrm{cap}(F)} = \inf\{E[\mu]\colon \mu \geqslant 0, \mathrm{supp}(\mu)\subset F, \mu(F)=1\}.$$

Proof: Apply Gauß' principle to $t\mu$, choose $t=\mathrm{cap}(F)$.

4 Heat Equation

- *Conservation of mass*: $\partial_t u + \mathrm{div}(\boldsymbol{v}) = 0$
- *Fick's law*: $\boldsymbol{v} = -\alpha^2\nabla u$.
- Together: $u_t = \Delta u$.
- Parabolic scaling invariance: $x\mapsto \lambda x$, $t\mapsto \lambda^2 t$.
- Use conservation of mass ($\partial_t \int u = 0$) to obtain the ansatz $u(x,t) = t^{-n/2} g(r\, t^{-1/2})$. Plug in heat equation to get the heat kernel

$$k(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

- Use

$$2\int_{y>a} e^{-y^2}\mathrm{d}y < 2\int_{y>a} \frac{y}{a} e^{-y^2} = \frac{e^{-a^2}}{a}.$$

and in-boxing the ball to show

$$\int_{|x|\geqslant\delta} k(x,t)\mathrm{d}x \to 0 \quad \text{as} \quad t\to 0.$$

- $u = k * f$ solves $u_t = \Delta u$ for $u\to f$ for $t\to 0$.
- *Tychonoff counterexample* for uniqueness:

$$u(x,t) = \sum_k g_k(t) x^{2k}$$

- *Widder's Theorem:* $u\geqslant 0 \Rightarrow$ uniqueness.
- *Heat ball:* $E(x,t,r) = \{k(x-y, t-r)\geqslant r^{-n}\}$.
- $V_T = U\times[0,T]$,
  $\partial_1 V_T =$ all except top "lid",
  $\partial_2 V_T =$ lid.
- *Mean Value Property:* $u\in C^2(V_T)$, $\partial_t u - \Delta u \leqslant 0$, $E(\ldots)\subset V_T$:

$$u(x,t) \leqslant \frac{1}{4r^n}\iint_{E(x,t,r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\mathrm{d}y\mathrm{d}s$$

  - Exists for heat spheres as well.

- *Converse:* Equality and $C^2(V_T)$ implies $\partial_t u = \Delta u$.

Proof: Let RHS$=\varphi(r)$. $\varphi(0) = u(x,t)$,

$$\varphi'(r) = -C\int (\partial_s u - \Delta u)\psi \,\mathrm{d}y\mathrm{d}s \geqslant 0$$

with $\{\psi \geqslant 0\} = E(\ldots)$.

- *Strong Maximum Principle:* U open, bounded, connected, $u \in C(\bar{V}_T)$ and satisfies MVI. Then

$$\max_{\bar{V}_T} u \leqslant \max_{\partial_1 V_T} u.$$

If max attained at $(x,t) \in V_T$, then u is constant in $\bar{V}_t$.
Proof: If max attained in interior, then $u = M$ on heat ball. Then a polygonal path reaches every point on V_T.

- *Temperatures are analytic:*
- *Green's functions for the heat equation:*
- *Strong Converse of MVP*.
Proof: Construct parallel solution by Green's functions. Conclude uniqueness by MVP.

4.1 Difference Schemes and Probabilistic Interpretation

- Work on a lattice.
- *Strong Maximum Principle* (subharmonic $\Rightarrow$ assume max M in interior $\Rightarrow M = u \leqslant E[x + h\omega] \leqslant M$.)
- Implies discrete Laplacian has trivial null-space $\Rightarrow \exists!$
- Allows *Discrete Poisson Integral Formula*. (by solving for δ on the boundary)
- *Markov property*: $E[X_{m+1}|X_1, \ldots, X_m] = E[X_{m+1}|X_m]$.
- *(Super)Martingale property*: u subharmonic $\Rightarrow$ $E[u(X_{m+1})|X_m] \geqslant u(X_m)$ (just like discrete SMP) [with X_m a random walk]
- *Strong Martingale Property:* m may be a stopping time.
- If M_U is first passage time to ∂U, then $u = E[f(x + W_{M_U})]$. (f=BC, u harmonic)

$$E[f(x + W_{M_U})] = \sum_{y \in \partial U_h} H(x,y) f(y) = \sum_{y \in \partial U_h} \underbrace{P(\text{hit } y)}_{H} f(y).$$

- *Method of relaxation*:

$$u^{(l+1)}(x) = \operatorname{avg}(u^{(l)} \text{ on pixels surrounding } x)$$

- *Brownian motion:* Same formula as above holds for continuous-time.
(Central Limit Theorem, path space version of it, $W_t \sim k(x, t/2)$. Cylinder sets. Convergence in weak-* topology. Law of iterated logarithm. Proof of CLT: Convolution of densities becomes multiplication after Fourier transform. Use independence. Done.)
- *Feynman-Kac formula:* $u_t = \frac{1}{2}\Delta u$ with IC f.

$$E(f(x + W_t)) = u(x,t)$$

- Implications on boundary regularity:
  - u *defined* by F-K is the Perron function
  - $y \in \partial U$ is regular iff $P(T_y = 0) = 1$ (BM immediately exits U.)
  - Littlewood's crocodile
  - Lebesgue's thorn

4.2 Hearing the shape of a drum

- *Spectral measure*:

$$A(\lambda) = \sum_{k=1}^{\infty} \mathbf{1}_{\lambda_k \leqslant \lambda}(\lambda).$$

- *Weyl's result*:

$$\lim_{\lambda \to \infty} \frac{A(\lambda)}{\lambda^{n/2}} = \frac{|U|}{(2\pi)^{n/2}\Gamma(n/2)}.$$

- *Kac's result:*

$$\lim_{t \to 0+} (2\pi t)^{n/2} \sum_{k=1} e^{-\lambda_k t} = (2\pi t)^{n/2} \int e^{-t\lambda} A(\mathrm{d}\lambda) = |U|.$$

  (Weyl $\Rightarrow$ Kac: Integrate by parts, rescale. Proof of Kac: represent Green's function in terms of eigenfunctions somehow.)

5 Wave equation

- $u_{tt} = c^2 u_{xx}$
- *D'Alembert's formula*:

$$u(x,t) = \frac{1}{2}\left[f(x + ct) + f(x - ct) + \frac{1}{c} \int_{x-ct}^{x+ct} g(y)\mathrm{d}y \right].$$

- *Characteristics*.
- *Parallelogram identity*:

$$u(\text{top}) + u(\text{bottom}) = u(\text{left}) + u(\text{right}).$$

- Good/bad BCs, Inflow/outflow. Domain of dependence. Method of reflection. Odd/even extension.
- *D'Alembertian*: $\Box u := u_{tt} - c^2 \Delta u = 0$. $u = f$, $u_t = g$.
- Fourier Analysis: $\hat{u}(\xi, t) = \hat{f}(\xi)\cos(c|\xi|t) + \hat{g}(\xi)\sin(c|\xi|t/|\xi|t) = \hat{f}(\xi)\cos(c|\xi|t) + \hat{g}(\xi)\partial_t \cos(c|\xi|t)$:

$$u(x,t) = \int_{\mathbb{R}^n} k(x - y, t) g(y)\mathrm{d}y + \partial_t \int_{\mathbb{R}^n} k(x - y, t) f(y)\mathrm{d}y$$

  Needs to coincide with solution formula.

- For $n = 3$, $k = t \cdot$ uniform measure on $\{|x| = ct\}$
- *Method of Spherical means:* Observe:

$$M_u(x,r) = \fint_{S(x,r)} u(y)\mathrm{d}S_y$$

  satisfies *Darboux's Equation:*

$$\Delta_x M_u = \text{“}\Delta_r\text{”} M_u = \left(\partial_{rr} - \frac{n-1}{r}\partial_r \right) M_u.$$

  Similarly, if u solves $u_{tt} = u_{xx}$, then M_u solves the *Euler-Poisson-Darboux equation*:

$$(M_u)_{tt} - \Delta_r M_u = 0.$$

  In 3D, this reduces the wave equation to $\partial_t^2(r\,M_u) = \partial_r^2(r\,M_u)$, which we can solve by D'Alembert's formula for all x. Then

$$u = \lim_{r \to 0} \frac{M_u}{r}.$$

-

$$\frac{1}{(2\pi)^{n/2}} \int_{|y| = ct} e^{-i\xi \cdot y}\mathrm{d}S_y = \frac{\sin(c|\xi|t)}{c|\xi|}.$$

- *Huygens' principle.*
- *Hadamard's method of descent:* Treat 2D equation as 3D equation, independent of third coordinate.
- *General solution for odd $n \geqslant 3$:* Assume $u'(0)=0$. Define

$$v(x,t) := \int k(s,t) u(x,s) \mathrm{d}s$$

  as a temporal heat kernel average. Oddly, $\partial_t v = \Delta_x v$. Solve this. Rewrite using spherical means. Change variables as $\lambda = 1/4t$ and invert using the Laplace transform

$$h^{\#}(\lambda) = \int_0^\infty e^{-\lambda\varphi} h(\varphi) \mathrm{d}\varphi.$$

- Uniqueness by energy norm.

6 Distributions/Fourier Transform

$U \subset \mathbb{R}^n$ open

- $\mathcal{D}(U) := C_c^\infty(U)$. $\varphi_k \to \varphi$ iff
  - $\exists$ fixed compact set F: $\mathrm{supp}(\varphi_k) \subset F$
  - $\forall \alpha$: $\sup_F |\partial^\alpha \varphi_k - \partial^\alpha \varphi| \to 0$.
- *Distribution:* $\mathcal{D}'(U)$
  - Convergence: $L_k \overset{\mathcal{D}}{\to} L \Leftrightarrow \forall \varphi \in \mathcal{D}(U)$: $(L_k, \varphi) \to (L, \varphi)$.
- Examples: $L^p_{\mathrm{loc}} \subset \mathcal{D}'(U)$. Aside: $L^p_{\mathrm{loc}} \subset L^q_{\mathrm{loc}}$ for $p \geqslant q$. (not for L^p), *Radon measure* (A Borel measure that is finite on compact sets.), δ function, Cauchy Principal value.
- *Derivative:* $(\partial^\alpha L, \varphi) = (-1)^{|\alpha|} (L, \partial^\alpha \varphi)$.
- Differentiation is continuous.
- *Partial differential operator*: $P = \sum_{|\alpha| \leqslant N} c_\alpha(x) \partial^\alpha$, adjoint, fundamental solution: $PK = \delta$.
- *Schwartz class:* $\mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$

$$\|\varphi\|_{\alpha,\beta} := \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty \quad \forall \alpha, \beta.$$

  A polynormed, metrizable space (Use $\sum 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{\|\cdot\|_{\alpha,\beta}}{1+\|\cdot\|_{\alpha,\beta}}$). Complete, too. (Arzelà-Ascoli).
- Examples:
  - $\mathcal{D} \subset \mathcal{S}$ (convergence carries over, too.)
  - $\exp(-|x|^2) \in \mathcal{S}$, but not $\in \mathcal{D}$.
- *Fourier Transform:*

$$\hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} \varphi(x) \mathrm{d}x$$

- Basic estimates:

$$\begin{aligned} \|\hat{\varphi}(\xi)\|_{L^\infty} &\leqslant C\|(1+|x|)^{n+1}\varphi(x)\|_{L^\infty} \leqslant C\|\varphi\|_{L^1} < \infty, \\ \|\partial_\xi^\beta \hat{\varphi}(\xi)\|_{L^\infty} &\leqslant C\|(1+|x|)^{n+1} x^\beta \varphi\|_{L^\infty} \\ \|\xi^\alpha \hat{\varphi}(\xi)\|_{L^\infty} &\leqslant C\|(1+|x|)^{n+1} \partial_x^\alpha \varphi\|_{L^\infty} \\ \|\hat{\varphi}\|_{\alpha,\beta} &\leqslant C\|(1+|x|)^{n+1} x^\beta \partial_x^\alpha \varphi\|_{L^\infty} \quad \Rightarrow \quad \hat{\varphi} \in C^\infty. \end{aligned}$$

- *Dilation:* $\sigma_\lambda \varphi(x) = \varphi(x/\lambda)$. $(\mathcal{F}\sigma_\lambda \varphi) = \lambda^n \sigma_{1/\lambda} \mathcal{F}\varphi$.

- *Translation:* $\tau_h\varphi(x) = \varphi(x-h)$. $(\mathcal{F}\tau_h\varphi) = e^{-ih\cdot\xi}\mathcal{F}\varphi$.
- *Inversion formula:*
$$\varphi(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\varphi}(\xi)\mathrm{d}\xi = \mathcal{F}^*\hat{\varphi} = \mathcal{F}\mathcal{R}\hat{\varphi},$$
where $\mathcal{R}\varphi(x) = \varphi(-x)$.
$\mathcal{F}$ is an isomorphism of $\mathcal{S}$, with $\mathcal{F}\mathcal{F}^* = \mathrm{Id}$.
Proof: Prove $(\mathcal{F}\mathcal{F}^* - \mathrm{Id})e^{-|x|^2} = 0$, then for dilations and translations, linear comb. of which are dense in $\mathcal{S}$. $\mathcal{F}$ is 1-1, $\mathcal{F}^*$ is onto, but $\mathcal{F}^* = \mathcal{R}\mathcal{F}$.
- $\mathcal{F}$ isometry of L^2, $\mathcal{F}$ continuous from L^p to L^q, where
$$\frac{1}{p} + \frac{1}{q} = 1, \quad p \in [1,2].$$
In particular $p = 1$, $q = \infty$.
Proof: Show $\mathcal{S}$ dense in L^p (see below), extend $\mathcal{F}$, use Plancherel for L^2.
- *Mollifier:* $\eta \in C_c^\infty$. $\int \eta = 1$. $\eta_N(x) := N^n\eta(Nx)$.
- $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. $(1 \leqslant p < \infty)$
Proof: $\|\eta_N * f - f\|_{L^p} \to 0$ holds for step functions. Step functions are dense in $L^p(\mathbb{R}^n)$.
$\|f * \eta_N\|_{L^p} \leqslant C\|f\|_{L^p}$ (Young's)
Pick g a step function such that $\|f-g\|_{L^p} < \varepsilon$. Now measure
$$\|f*\eta_N - f\|_{L^p} = \|f*\eta_N - g*\eta_N + g*\eta_N - g + g - f\|_{L^p}.$$
- $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}$.
Proof: Smooth cutoff.
- *Plancherel's Theorem:* $(\mathcal{F}f, \mathcal{F}g)_{L^2} = (f,g)_{L^2}$.
Proof: by Fubini.
- $\mathcal{F}: L^1(\mathbb{R}^n) \to \dot{C}(\mathbb{R}^n)$, with $\dot{C} := \{h: \mathbb{R}^n \to \mathbb{R}: h(x) \to 0\,(x \to \infty)\}$.
Proof: $\mathcal{S}$ is dense in L^1. Well-defined: Take $\varphi_k, \psi_k \to f \in L^1$, show $\mathcal{F}\varphi_k - \mathcal{F}\psi_k \to 0$ in L^∞.
Goes to $\dot{C}$: unproven.
- *Linear operator of type* (r,s):
$$\|K\varphi\|_{L^s} \leqslant C(r,s)\|\varphi\|_{L^r}.$$
$\mathcal{F}$ is of type $(1,\infty)$ and $(2,2)$.
- *Riesz-Thorin Convexity Theorem:* $\mathcal{F}$ of type (r_0,s_0) and (r_1,s_1)
$$\begin{aligned}\frac{1}{r} &= \frac{\theta}{r_0} + \frac{1-\theta}{r_1}\\ \frac{1}{s} &= \frac{\theta}{s_0} + \frac{1-\theta}{s_1}\end{aligned}$$
Then $\mathcal{F}$ of type (r,s) for $\theta \in [0,1]$.

6.1 Tempered Distributions

- *Tempered Distributions:* $\mathcal{S}'$, convergence as in $\mathcal{D}'$. $\mathcal{D} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{D}'$.
Examples: L^1 functions, $e^{|x|^2}$ not, $e^{-|x|^2}$, $\left\|(1+|x|^2)^{-M}f\right\|_{L^1} < \infty$.
- A tempered distribution is no worse than a certain derivative coupled with a monomial multiplication.
$L \in \mathcal{S}' \Rightarrow \exists C, N \forall \varphi \in \mathcal{S}: |(L,\varphi)| \leqslant \sum_{|\alpha|,|\beta| \leqslant N} \left\|x^\alpha\partial^\beta\varphi\right\|_{L^\infty}$ (continuity).
- $(\eta * L, \varphi) = (L, (\mathcal{R}\eta) * \varphi)$ for $L \in D'$, $\mathcal{R}$ is reflection and η a mollifier
- $\eta * L$ is a C^∞ function, namely $\gamma(x) = (L, \tau_x\mathcal{R}\eta)$, where $\tau_x f(y) = (y-x)$.

Proof: 1. γ maps to $\mathbb{R}$. 2. γ sequentially continuous. 3. $\gamma \in C^1$ (FD). 4. $\gamma \in C^\infty$ (induction). 5. $(\eta * L, \varphi) = (\gamma, \varphi)$ (Riemann sums).

- $\mathcal{D}$ is dense in $\mathcal{D}'$.
  Proof: $\chi_m := \mathbf{1}_{[-m,m]}$. Fix $L \in \mathcal{D}'$, $L_m := \chi_m(\eta_m * L) \in D \to L$ in $\mathcal{D}'$.
- $\mathcal{S}$ is dense in $\mathcal{S}'$.
  (because $\mathcal{D}$ is already dense in $\mathcal{D}'$.)
- *Transpose* $K^t: \mathcal{S} \to \mathcal{S}$ for $K: \mathcal{S} \to \mathcal{S}$ as by $\left(K^t L, \varphi\right) := (L, K\varphi)$.
- $K: \mathcal{S} \to \mathcal{S}$ linear and continuous. $K^t|_{\mathcal{S}}$ continuous. $\exists$!unique, continuous extension of K^t onto $\mathcal{S}'$.
- $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$ continuous.
- $\mathcal{F}\delta = 1/(2\pi)^{n/2}$.
- $0 < \beta < n$, $C_\beta = \Gamma((n-\beta)/2)$
  $$\mathcal{F}(C_\beta |x|^{-\beta}) = C_{n-\beta} |x|^{-(n-\beta)}.$$
  Use this to solve Laplace's equation.

7 Hyperbolic Equations

- General constant coefficient problem. $P(D, \tau) = \tau^m + \tau^{m-1} P_1(D) + \cdots + P_m(D)$
- *Duhamel's principle:* Solve $P(D, \tau) u = f$ by solving the *standard problem* $P(D, \tau) u_s = 0$, $u_s(0) = 0$, $\partial_t^{m-1} u_s(0) = g$ and finding
  $$u(x, t) = \int_0^t u_s \mathrm{d}s.$$
- Treat remaining ICs by solving standard problems for $\tau^{m-1} P_1$, ..., $\tau^0 P_m$, each time adding to the right hand side, which can finally be killed with the above approach.
- Fourier-transforms to $P(i\xi, \tau)\hat{u} = 0$, with $\tau = \partial_t$.
  Initial conditions $\tau^{0 \ldots m-2} \hat{u}(\xi, 0)$, $\tau^{m-1} \hat{u}(\xi, 0)$.
- Representation of the solution:
  $$\begin{aligned} Z(\xi, t) &= \frac{1}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda \\ P(i\xi, \tau) Z &= \frac{1}{2\pi} \int_\Gamma P(i\xi, i\lambda) \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} \mathrm{d}\lambda = \frac{1}{2\pi} \int_\Gamma e^{i\lambda t} \mathrm{d}\lambda = 0, \end{aligned}$$
  where Γ is a path around the roots.
- Classical solution requires $u \in C^m$. Requires $\forall T \exists C_T, N$:
  $$|\tau^k Z(\xi, t)| \leqslant C_T (1 + |\xi|)^N.$$
- *Hyperbolicity:* A standard problem is *hyperbolic*:$\Leftrightarrow \exists$a C^m solution for all $g \in \mathcal{S}(\mathbb{R}^n)$.
- *Gårding's Criterion:* It's hyperbolic iff $\exists c \in \mathbb{R}$: $P(i\xi, i\lambda) \neq 0$ for all ξ and $\operatorname{Im} \lambda \leqslant -c$.
  Proof: Estimate around in the above representation for Z.
- *Paley-Wiener Theorem:* $g \in L^1 \Rightarrow \hat{g}$ entire.

8 Conservation Laws

- $u_t + f(u)_x = 0$.

Why are they called called conservation laws?

$$\frac{\mathrm{d}}{\mathrm{d}t}\int u = \int u_t = \int f(u)_x = f(b) - f(a) \to 0.$$

- *Inviscid Burgers' Equation:* $u_t + (u^2)_x = 0$.
- *Characteristics:* Assume $u = u(x(t), t)$,

  $$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial t}$$

  Compare shape with

  $$0 = u_x f'(u) + u_t,$$

  obtain $\mathrm{d}x/\mathrm{d}t = f'(u)$.
- *Weak solution*: slap test function onto equation, integrate by parts.
- *Rankine-Hugoniot*:

  $$\text{shock speed} = \frac{[\![f(u)]\!]}{[\![u]\!]}$$

  Apply weak solution formula across a jump, consider normal geometrically to obtain speed.
- *Riemann problem*: Jump IC. $\to$ non-uniqueness of the weak solution for jump up: rarefaction wave or shock with correct speed?
- *Hopf's treatment of Burger's Equation:*
  - Add viscosity to get $u_t + (u^2/2)_x = \varepsilon u_{xx}$.
  - Put U as an antiderivative of u.
  - Gives Hamilton-Jacobi PDE $U_t + U_x^2/2 = \varepsilon U_{xx}$.
  - Now try to rewrite that into a linear equation, by assuming $\psi = \psi(u)$. Yields ODE $C\varphi'' + C\varphi' = 0$, solution $\psi = \exp(-U/2\varepsilon)$.
  - This gives the heat equation $\psi_t = \varepsilon\psi_{xx}$.
  - $$u = 2\varepsilon\frac{\psi_x}{\psi} = \frac{\int \frac{x-y}{t}\exp(-G/2\varepsilon)\mathrm{d}y}{\int \exp(-G/2\varepsilon)\mathrm{d}y} = \frac{x}{t} - \frac{\langle y\rangle}{t} \to \frac{x}{t} - \frac{\operatorname{argmin} G}{t}$$

    with $G = (x-y)^2/2t + U_0$.
- $a_- = \inf \operatorname{argmin} G$, $a_+ = \sup \operatorname{argmin} G$.
- *Properties:* well-defined, increasing, $a_+(\leftarrow) \leqslant a_-(\rightarrow)$, a_- left-continuous, a_+ right-continuous, go to $\pm\infty$. Equal except for a countable set of shocks.
- *Hopf's theorem:*

  $$\frac{x-a_+}{t} \leqslant \liminf_{\varepsilon\to 0} u^\varepsilon \leqslant \limsup_{\varepsilon\to 0} u^\varepsilon \leqslant \frac{x-a_-}{t}$$

- $u_0 \in \mathrm{BC}$ (bounded, continuous) $\Rightarrow u(\cdot, t) \in \mathrm{BV}_{\mathrm{loc}}$. Globally BV?
  Proof: x, a_+, a_- are increasing $\Rightarrow$ differences in $\mathrm{BV}_{\mathrm{loc}}$.
- Vanishing viscosity solutions are weak solutions.
  Proof: Pass to vanishing viscosity under integral using DCT and boundedness.
- Cole-Hopf solutions produce rarefaction x/t for jump up, shock for jump down.
- More properties:
  - $\lim_{\varepsilon\to 0} u^\varepsilon$ exists except for a countable set. $u = \lim u^\varepsilon \in \mathrm{BV}_{\mathrm{loc}}$ with left and right limits.
    Proof: u is a difference of increasing functions.
  - *Lax-Oleinik entropy condition:* $u(x_-, t) > u(x_+, t)$ at jumps.

"Characteristics never leave a shock."
Proof: Travelling waves for Burgers with viscosity only exist for $u_- > u_+$.

- ○ x a shock location:

$$\begin{aligned}\text{shock speed} &= \frac{[\![f(u)]\!]}{u} = \frac{1}{2}(u_+ + u_-),\\ \text{shock speed} = \frac{1}{2}(u(x_-,t)+u(x_+,t)) &= \fint_{a_-}^{a_+} u_0(y)\mathrm{d}y,\\ (a_+ - a_-)\text{shock speed} &= \int_{a_-}^{a_+} u_0(y)\mathrm{d}y\end{aligned}$$

The last equation here is a momentum conservation equality.
Proof: $G(a^+) = G(a^-)$.

- *Entropy/entropy-flux pair*: Φ, Ψ: $\mathbb{R}^m \to \mathbb{R}$ smooth are an e/ef pair for $u_t + f(u)_x = 0$: $\Leftrightarrow$ Φ convex, $\Phi' f' = \Psi'$. Then $\Phi(u)_t + \Psi(u)_x = 0$ for perfectly smooth solutions, otherwise $\Phi(u_t) + \Psi(u)_x \leqslant 0$ in the distributional sense, which means

$$\int_0^\infty \int_{-\infty}^\infty \Phi(u)v_t + \Psi(u)v_x \mathrm{d}x\mathrm{d}t \geqslant 0.$$

for smooth non-negative v.

- By the vanishing viscosity method, we get an entropy solution.
Proof: Multiply the viscosity-added c.law by Φ'. Use chain rule on $\Phi(u^\varepsilon)_{xx}$. Use convexity of Φ to show one term involving φ'' non-negative. Multiply by a non-negative smooth function, let $\varepsilon \to 0$ to obtain entropy inequality.

- *Entropy solution*: u is an entropy solution of a c.law if u is a weak solution that satisfies the entropy condition for every e/ef pair.

- *Dissipation measure:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (u^\varepsilon)^2 = -2\varepsilon \int (u_x^\varepsilon)^2.$$

Assuming a traveling wave solution of the form

$$u^\varepsilon = v\left(\frac{x - ct}{\varepsilon}\right),$$

we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (u^\varepsilon)^2 = \frac{(u_- - u_+)^3}{6}.$$

- *Kružkov's Uniqueness Theorem:* L^∞ Entropy solutions u, v, S_t cuts of the event cone (given by max. speed $c^* = \max_{\text{range}\, u} |f'|$. Then for $t_1 < t_2$

$$\int_{S_{t_2}} |u - v| \leqslant \int_{S_{t_1}} |u - v|.$$

Proof: Doubling trick, clever choice of test functions.
Implies uniqueness.

9 Hamilton-Jacobi Equations

- $u_t + H(Du, x) = 0$.
- Example: Curve evolving with normal velocity: $u_t + \sqrt{1 + |D_x u|^2} = 0$.
- Non-Example: Motion by mean curvature $u_t = u_{xx}/(1 + u_x)^2$ (parabolic).
- Example: Substitute $U = \int u$ in conservation laws.

- PDE is infinitely-many-particle limit of Hamilton ODE

$$\begin{aligned}\dot{x} &= \partial_p H(p,x)\\ \dot{p} &= -\partial_x H(p,x),\end{aligned}$$

  which coincides with characteristic equation of PDE.

- Mechanics motivation:

  ◦ $L(q,x) = T - V$

  ◦ Lagrange's Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial q}\right) = \frac{\partial L}{\partial x}.$$

  Way to see this: If RHS $= 0$, then L symmetric in x, so LHS becomes conserved. (Noether's theorem.)
  Equivalent to Hamilton's ODE (Proof: $H = \max_q (q\,p - L(x,q,t))$, where $q = q(x,p,t)$ is the solution of $p = \partial_q L(v)$.

  ◦ Action, given path $x(t)$:

$$S(x) = \int_0^t L(\dot{x}, x, t)\mathrm{d}t$$

  ◦ Principle of least action: $\min S \Leftrightarrow$ Lagrange's Equation.
  Proof: $u + \varepsilon v$, derivative by ε, the usual.

  ◦ Generalized momentum: $p = \partial_q L$. Assumed solvable for q.

  ◦ Hamiltonian: $H = T + V = p \cdot q - L = 2T - (T - V) = T + V$.

  ◦ *Legendre transform:* More general way of obtaining H. Assume $L(q)$ (dropping dependencies!) convex, $\lim_{|q|\to\infty} L(q)/|q| = \infty$. Then

$$H(p) = L^*(p) = \sup_q \{p \cdot q - L(q)\}.$$

  Solved when $p = \partial_q L$, but in a more general sense.
  Duality: Edge $\leftrightarrow$ Corner. Subdifferentials.

  ◦ L convex $\Rightarrow L^{**} = L$.
  Proof: Prove convexity and superlinearity of L^*. Use symmetry

$$H(p) + L(q) \geqslant p \cdot q$$

  to prove two sides of the equality $H^* = L$.

- *Hopf-Lax formula*: g is IC

$$u(x,t) = \inf\left\{\int L(\dot{x})\mathrm{d}x + g(y), x(0) = y, x(t) = x\right\} = \min\left\{t\,L\left(\frac{x-y}{t}\right) + g(y)\right\}.$$

  Proof: Inf bounded above by straight-line characteristic. Lower bound works by Jensen's inequality.

- *Semigroup Property*.
  Proof: Always pick particular solutions, prove both sides of the inequality.

- u defined by Hopf-Lax is Lipschitz if g is Lipschitz.
  Proof: Lipschitzicity for given t is immediate (pick good z). Transform problem to comparison with $t = 0$ by semigroup property. Temporal estimate is screwy, involves special choices in inf.

- u by Hopf-Lax is differentiable a.e. and satisfies the H-J PDE where it is.
  Proof: Rademacher's Theorem. Prove $u_t + H(Du) \leqslant 0$ for forward in time by taking increments $\to 0$, using inequality with Legendre transform.

- Lipschitz+Differentiable solution a.e. is not sufficient for uniqueness. (45-degree angle trough vs. 90-degree trough)
- $f\colon \mathbb{R}^n \to \mathbb{R}$ *semiconcave* if
$$f(x+z) - 2f(x) + f(x-z) \leqslant C|z|^2$$
for some z.
$\Leftrightarrow f(z) - C/2|z|^2$ is concave.
$\Leftrightarrow$"can be forced into concavity by subtracting a parabola."
$\Leftarrow C^2$ and bounded second derivatives implies semiconcavity.
- g semiconcave $\Rightarrow u$ semiconcave.
Clever choice of test locations in Hopf-Lax.
- $H\colon \mathbb{R}^n \to \mathbb{R}$ *uniformly convex*: $\Leftrightarrow$
$$\sum_{i,j} H_{p_i p_j} \xi_i \xi_j \geqslant j|\xi|^2.$$
- If H uniformly convex. Then u is semiconcave (indep. of initial data)
Proof: Taylor, mess about with Hopf-Lax.
- Now $H(p) \to H(p, x)$ nonconvex.
- *Vanishing Viscosity Method:* Use $u_t + H(Du, x) = \varepsilon \Delta u$. Locally uniform convergence follows from Arzelà-Ascoli.
- u is a *viscosity solution*: $\Leftrightarrow u = g$ on $\mathbb{R}^n \times \{t = 0\}$, for each $v \in C^\infty(\mathbb{R}^n \times (0, \infty))$
$$u - v \text{ has a local maximum at } (x_0, t_0) \Rightarrow v_t(x_0, t_0) + H(Dv(x_0, t_0)) \leqslant 0 \text{ (and min} \to \geqslant).$$
- If u is a *vanishing* viscosity solution, then it is a viscosity solution.
Proof: Convergence is locally uniform as $\varepsilon_j \to 0$. Thus for each fixed ball around a local strict maximum in $u - v$, a local maximum in $u^\varepsilon - v$ exists if ε is small enough. There, $v_x = u_x^\varepsilon$ and $v_t = u_t^\varepsilon$ and $-\Delta u^\varepsilon \geqslant -\Delta v$. $v_t + H(Dv) \leqslant 0$ follows. Generalize to non-strict maxima by adding parabolas.
- A classical solution of a H-J PDE is a viscosity solution.
Proof: Maximum of $u - v \Rightarrow$ derivatives are equal $\Rightarrow$ PDE.
- *Touching by C^1 function*: u continuous. u differentiable at x_0. Then $\exists v \in C^1$: $v(x_0) = u(x_0)$, $u - v$ has a strict local max.
- u viscosity solution $\Rightarrow u$ satisfies H-J wherever it is differentiable
Proof: Mollify touching function, $u - v^\varepsilon$ maintains strict max., verify definition of Viscosity solution. (Mollification necessary because test functions are required to be C^∞.)
- *Uniqueness:* $H \in \mathrm{Lip}_p(C) \cap \mathrm{Lip}_x(C1 + |p|) \Rightarrow$ uniqueness.
Proof: doubling trick again.

10 Sobolev Spaces

$1 \leqslant p < \infty$.

- $\|u\|_{k,p;\Omega} = \sum_{|\alpha| \leqslant k} \|D^\alpha u\|_p$.
- $W^{k,p}(\Omega) := \{u \in \mathcal{D}'(\Omega) \colon D^\alpha u \in L^p(\Omega), |\alpha| \leqslant k\}$ Banach space.
- $W_0^{k,p}(\Omega) := \mathrm{cl}(\mathcal{D}(\Omega), \|\cdot\|_{k,p;\Omega})$.
- $u \in W^{l,p}(\Omega)$. $\Omega' \subset\subset \Omega$ open $\Rightarrow \exists u_k \in C_c^\infty(\Omega')$: $\|u_k - u\|_{l,p;\Omega'} \to 0$.
Proof: Mollification, throw derivatives onto u by integration by parts.
- $u \in W^{l,p}(\Omega)$, Ω bounded $\Rightarrow \exists u_k \in C^\infty(\Omega) \cap W^{l,p}(\Omega)$: $\|u_k - u\|_{l,p;\Omega} \to 0$.

Proof: Exhaust Ω by $U_k := \{\mathrm{dist}(x, \partial U) > 1/k\}$. Consider smooth partition of unity ζ_i subordinate to $V_i := \Omega_{i+3} \setminus \bar{\Omega}_{i+1}$. $u_i := \eta_{\varepsilon_i} * (\zeta_i u)$ s.t. $\|u_i - \zeta_i u\|_{l,p} < \delta 2^{-i-1}$. Give one more set of wiggle room on each side for mollification. $v := \sum \zeta_i u_i \in C^\infty$ because there's only a finite number of terms for fixed point/set. Then estimate $\|u - v\|_{l,p}$.

- Typical idea: Consider
$$f^*(x) = \lim_{r \to 0} \fint_{B(x,r)} f(y) \mathrm{d}y.$$
- $u \in W^{1,p}(\Omega)$, $\Omega' \subset\subset \Omega$. Then
  - There exists a representative on Ω' that is absolutely continuous on a line and whose classical derivative agrees a.e. with the weak one.
  - If the above is true of a function, then $u \in W^{1,p}(\Omega)$.

  Proof: WLOG $p = 1$ (Jensen). WTF?
  Consequences: $W^{1,p}$ closed wrt. max, min, abs. value, $\cdot^+$. Ω connected, $Du = 0 \Rightarrow u$ constant.

10.1 Campanato

- *Oscillation:*
$$\mathrm{osc}_U = \sup_{x,y \in U} |u(x) - u(y)|.$$
- $C^{0,\alpha} := \{|u(x) - u(y)| \leqslant C|x - y|^\alpha\}$. $\|u\|_{C^{0,\alpha}} := \|u\|_{C(\bar{U})} + \sup_{x \neq y} |u(x) - u(y)|/|x - y|^\alpha$.
- $C^{k,\alpha} := D^\alpha \in C^{0,\alpha}$. Norm: sum over multi-indices.
- *Campanato's Inequality:* $u \in L^1_{\mathrm{loc}}(\Omega)$, $0 < \alpha \leqslant 1$, $\exists M > 0$:
$$\fint_B |u(x) - \bar{u}_B(x)| \mathrm{d}x \leqslant M r^\alpha.$$
  Then $u \in C^{0,\alpha}(\Omega)$ and $\mathrm{osc}_{B(x,r/2)} u \leqslant C M r^\alpha$. $\bar{u}_B$ is the mean over B.
  Proof: x a Lebesgue point of u, $B(x, r/2) \subset B(z, r)$. Then $|\bar{u}_{B(x,r/2)} - \bar{u}_{B(z,r)}| \leqslant 2^n M r^\alpha$. Iteration via geometric series and Lebesgue-pointy-ness yields
$$|u(x) - \bar{u}_{B(z,r)}| \leqslant C(n, \alpha) M r^\alpha.$$
  For two Lebesgue points,
$$|u(x) - u(y)| \leqslant |u(x) - \bar{u}_{B(z,r)}| + |\bar{u}_{B(z,r)} - u(y)| \leqslant C(n, \alpha) M r^\alpha.$$

10.2 Sobolev

- *Gagliardo-Nirenberg-Sobolev*: $u \in C^1_c(\mathbb{R}^n)$, $1 \leqslant p < n \Rightarrow$
$$\|u\|_{p^*} \leqslant C \|Du\|_p,$$
  where
$$\frac{1}{p^*} + \frac{1}{n} = \frac{1}{p} \quad \Rightarrow \quad p^* > p.$$
- Considering what happens when you scale functions $u \to u_\lambda(x) := u(\lambda x)$, these exponents are the only ones possible.
- If we choose $p = 1$, then the best constant comes to light by choosing $u = \mathbf{1}_{B(0,1)}$, giving the isoperimetric inequality.
- Proof: Suppose $p = 1$ at first. Compact support $\Rightarrow$
$$u(x) \leqslant \int_{-\infty}^{\infty} |Du(x \ldots x, y_i, x, \ldots, x)| \mathrm{d}y_i \quad (i = 1, \ldots, n).$$

Then

$$|u(x)|^{n/(n-1)} \leqslant \left(\prod_i \int \ldots \mathrm{d}y_i\right)^{1/(n-1)}.$$

Integrating this gives

$$\int |u|^{n/(n-1)}\mathrm{d}x_1 \leqslant \left(\int |Du|\mathrm{d}x_1\right)^{1/(n-1)} \left(\prod_{i=2} \iint |Du|\mathrm{d}x_1\mathrm{d}y_i\right)^{1/(n-1)}$$

by pulling out an independent part and using generalized Hölder. Then iterate the same trick. To obtain for general p, use on $v = |u|^\gamma$ with suitable γ.

10.3 Poincaré and Morrey

- *Riesz potential*: $0 < \alpha < n$

$$I_\alpha(x) = |x|^{\alpha - n} \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

- $\|I_1 * f\|_{L^p} \leqslant C\|f\|_{L^p}$.
- *Poincaré's Inequality*: Ω convex, $|\Omega| < \infty$, $d = \mathrm{diam}(\Omega)$, $u \in W^{1,p}(\Omega)$. Then

$$\left(⨍_\Omega |u(x) - \bar{u}_\Omega|^p\right)^{1/p} \leqslant C\,d\left(⨍_\Omega |Du|^p\right)^{1/p}.$$

Proof: Use calculus to derive

$$|u(x) - \bar{u}\,| \leqslant \frac{d^n}{n} ⨍_\Omega \frac{|Du(y)|}{|x-y|^{n-1}}\mathrm{d}y.$$

Then use potential estimate.

- *Morrey's Inequality:* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $0 < \alpha \leqslant 1$. If $\exists M > 0$ with

$$\int_{B(x,r)} |Du| \leqslant M r^{n-1+\alpha},$$

for all $B(x,r) \subset \Omega$. Then $u \in C^{0,\alpha}(\Omega)$ and $\mathrm{osc}_{B(x,r)} u \leqslant C M r^\alpha$.

- Morrey=Poincaré+Campanato in $W^{1,1}$.
- *More general Morrey:* $u \in W^{1,p}(\mathbb{R}^n)$, $n < p \leqslant \infty$. Then $u \in C^{0,1-n/p}_{\mathrm{loc}}(\mathbb{R}^n)$ and

$$\mathrm{osc}_{B(x,r)} u \leqslant r^{1-n/p}\|Du\|_{L^p}.$$

If $p = \infty$, u is locally Lipschitz.

Proof: Use Jensen $(\cdot)^{p\cdot\frac{1}{p}}$ on Poincaré's RHS. Then apply Campanato.

10.4 BMO

- *BMO seminorm*:

$$[u]_{\mathrm{BMO}} := \sup_B ⨍_B |u - \bar{u}_B|\mathrm{d}x$$

- $\mathrm{BMO} := \{[u]_{\mathrm{BMO}} < \infty\}$.
- *John-Nirenberg*: $W^{1,n}(\mathbb{R}^n)(\cap L^1(\mathbb{R}^n)) \subset \mathrm{BMO}(\mathbb{R}^n)$.
  Proof: Poincaré-then-Jensen.
- For a compact domain, $L^p \subset L^\infty \subset \mathrm{BMO}$.

10.5 Imbeddings

- *Imbedding* $B_1 \to B_2$: $\exists$continuous, linear, injective map.

- $W^{1,p}(\mathbb{R}^n) \to L^{p^*}$ for $1 \leqslant p < n$ (Sobolev inequality)
- $W^{1,p}(\mathbb{R}^n) \to \mathrm{BMO}$ for $p = n$
- $W^{1,p}(\mathbb{R}^n) \to C^{0,1-n/p}_{\mathrm{loc}}$ (Morrey)

Ω bounded now.

- $W^{1,p}(\Omega) \to L^q(\Omega)$ for $1 < p < n$ and $1 \leqslant q < p^*$. Proof: Hölder-then-Sobolev:

$$\|u\|_{L^q} \leqslant \|u\|_{L^{p^*}} |\Omega|^{1-q/p^*} \leqslant \|Du\|_{W^{1,p}}.$$

- $W^{1,p}_0(\Omega) \to C^{0,1-n/p}(\bar{\Omega})$ for $n < p \leqslant \infty$.
- *Compact imbedding* $B_1 \hookrightarrow B_2$: The image of every bounded set in B_1 is precompact in B_2. (precompact: has compact closure)
- Rellich-Kondrachev:
  - $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 < p < n$ and $1 \leqslant q < p^*$.
    In Evans, we need $\partial U \in C^1$. Our notes do not.
    Proof:
    - Grab a $W^{1,p}$-bounded sequence u_m.
    - Mollify it to u_m^ε
    - Use an ε-derivative trick to show $\|u_m^\varepsilon - u_m\|_{L^1} \leqslant \varepsilon \|Du_m\|_{L^p} \to 0$
    - Interpolation inequality for L^p: $\|u_m^\varepsilon - u_m\|_{L^q} \leqslant \|u_m^\varepsilon - u_m\|_{L^1}^\theta \|u_m^\varepsilon - u_m\|_{L^{p^*}} \to 0$, also using GNS.
    - For fixed ε, u_m^ε is bounded and equicontinuous (directly mess with convolution).
    - Use Arzelà-Ascoli and a diagonal argument to finish off.
  - $W^{1,p}_0(\Omega) \hookrightarrow C^0(\bar{\Omega}) \subset L^p(\Omega)$ for $n < p \leqslant \infty$.
    Proof: Morrey's Inequality, then Arzelà-Ascoli.

11 Scalar Elliptic Equations

- $Lu = \mathrm{div}(A\,Du + bu) + c \cdot Du + d\,u$.
- Motivation: Calculus of Variations.
- *Weak Formulation*: $u \in W^{1,2}(\Omega)$, $v \in C^1_c(\Omega)$

$$B[u,v] := \int_\Omega (Dv^T A\,Du + b \cdot Dv\,u) - (c \cdot Du + d\,u)v\,\mathrm{d}x.$$

- *Generalized Dirichlet Problem*: $Lu = g + \mathrm{div}\, f$ on Ω, $u = \varphi$ on $\partial\Omega$, i.e. $B[u,v] = F(v)$ with

$$F(v) := \int_\Omega Dv \cdot f - g\,v\,d\mathrm{d}x.$$

- Assumptions:

  (*E*₁). *Strict ellipticity:* $\exists \lambda > 0$: $\xi^T A \xi \geqslant \lambda |\xi|^2$

  (*E*₂). *Boundedness:* $A, b, c, d \in L^\infty$, i.e. $\left\|\mathrm{Tr}(A^T A)\right\|_{L^\infty} \leqslant \Lambda^2$, $\frac{1}{\lambda^2}(\|b\|_\infty + \|c\|_\infty) + \frac{1}{\lambda}(\|d\|_\infty) \leqslant \nu$.

  (*E*₃). $\mathrm{div}\, b + d \leqslant 0$ weakly, i.e.

$$\int_\Omega d\,v - b \cdot Dv\mathrm{d}x \leqslant 0$$

  for $v \in C^1_c(\Omega)$, $v \geqslant 0$.

- *"$\leqslant$" on the boundary*: $u \leqslant v \Leftrightarrow (u-v) \leqslant 0 :\Leftrightarrow (u-v)^+ \in W_0^{1,2}(\Omega)$.
- *"sup" on the boundary*: $\sup_{\partial\Omega} u = \inf \{k \in \mathbb{R}: u \leqslant k \text{ on } \partial\Omega\}$.
- *u is a subsolution*: $\Leftrightarrow B[u,v] \leqslant F(v) \Leftrightarrow L u \geqslant g + \operatorname{div} f$.
- *Non-divergence form:*

  $$0 = A\, D^2 U + b \cdot D u + d\, u$$

  (Not equivalent!)
- *Classical Maximum Principle:* Holds if $d \leqslant 0$.
- *Weak Maximum Principle:* $L u \geqslant 0 \Leftrightarrow B[u, v] \leqslant 0$ for $v \geqslant 0$ and (E_1), (E_2), (E_3). Then $\sup_\Omega u \leqslant \sup_{\partial\Omega} u^+$.
  Proof:
  - Use $B[u,v] \leqslant 0$ for $v \geqslant 0$ and (E_3) to establish

    $$\int D v^T A\, D v - (b+c) D u \cdot v \leqslant \int d(u\, v) - b \cdot D(u\, v) \leqslant 0.$$

    Note that $u\, v$ is the new test function in (E_3). Consequently

    $$\int D v^T A\, D v \leqslant \int (b+c) D u \cdot v.$$

  - Suppose $l = \sup_{\partial\Omega} u \leqslant k < \sup_\Omega u$. Set $\Gamma := \{u > k\}$ and achieve a $\|D v\|_{L^2} \leqslant C \|v\|_{L^2}$ estimate by using ellipticity, the above and boundedness. Use the Sobolev inequality to get $\|v\|_{L^{2^*}} \leqslant \cdots \leqslant |\Gamma|^{1/n} \|v\|_{L^{2^*}}$, and so $|\Gamma| > 0$ independently of k. Let $k \to \sup_\Omega$ to obtain a contradiction. (Note $\sup_\Omega < \infty$ because $u \in W^{1,2}(\Omega)$.)

  Remarks:
  - Implies uniqueness.
  - No assumptions on boundedness, smoothness or connectedness of Ω.
- Implies uniqueness.

11.1 Existence Theory

- *Existence:* Ω bounded, (E_1), (E_2), (E_3). Then $\exists!$ solution of the generalized Dirichlet problem.
  - Reduce BC to $H_0^{1,2}$ by subtracting arbitrary function and handling RHS.
  - Prove coercivity estimate

    $$B[u,u] \geqslant \frac{\lambda}{2} \int_\Omega |D u|^2 \mathrm{d}x - \lambda \nu^2 \int_\Omega |u|^2 \mathrm{d}x.$$

    (Uses: (E_1), (E_2), $2\, a\, b \leqslant \lambda a^2 + b^2/\lambda$.
    (In Evans, Poincaré enters here. How?)
    (For Δ, Poincaré suffices to show coercivity.)
  - $\mathrm{Id}: W_0^{1,2} \to (W_0^{1,2})^*$ is compact.

    $$\mathrm{Id} = \underbrace{(L^2 \to \mathcal{H}^*)}_{\text{continous}} \circ \underbrace{(\mathcal{H} \to L^2)}_{\text{compact}}.$$

  - $L_\sigma := L - \sigma \mathrm{Id}$. ($L \approx \Delta$ has negative eigenvalues already. But they might be pushed upward by the first- and zeroth-order junk. So we might have to make them even more negative to succeed.)
  - $\to B_\sigma[u,v] = B[u,v] + \sigma (u,v)_{L^2}$, coercivity is maintained.

- ○ Lax-Milgram shows existence of inverse L_σ^{-1} for the not-so-bad operator L_σ.
- ○ Start with $Lu = g + \operatorname{div} f$, introduce L_σ, multiply by L_σ^{-1} and see what happens.
- ○ Weak maximum principle provides uniqueness for L, so that the Fredholm alternative provides existence when combined with Rellich.

11.2 Regularity

- Assumptions:
  - ○ (R_1): (E_1), (E_2).
  - ○ (R_2): $f \in L^q(\Omega)$, $g \in L^{q/2}$, $q > n$.
- (R_1), $Lu = g$. A, b Lipschitz. Then for $\Omega' \subset\subset \Omega$ we have

$$\|u\|_{W^{2,2}(\Omega')} \leqslant C\Big(\|u\|_{W^{1,2}(\Omega)} + \|g\|_{L^2(\Omega)}\Big).$$

  Proof:
  - ○ Finite Differences.

11.3 Harnack Inequality Stuff

- *(Ladyzhenskaya/Uraltseva)*: (R_1), (R_2). $u \in W^{1,2}$ a subsolution, $u \leqslant 0$ on $\partial\Omega$. Then:

$$\sup_\Omega u \leqslant C\Big(\|u^+\|_{L^2(\Omega)} + k\Big),$$

  where

$$k = \frac{1}{\lambda}\Big(\|f\|_{L^q} + \|g\|_{L^{q/2}}\Big).$$

  Proof:
  - ○
- *Local Boundedness*: (R_1), (R_2). $u \in W^{1,2}$ a subsolution. Then:

$$\sup_{B(y,R)} u \leqslant C\Big(R^{-n/p}\|u^+\|_{L^2(\Omega)} + k(R)\Big),$$

  where

$$k(R) = \frac{R^{1-n/q}}{\lambda}\Big(\|f\|_{L^q} + R^{1-n/q}\|g\|_{L^{q/2}}\Big).$$

- *Weak Harnack Inequality*: (R_1), (R_2), $u \in W^{1,2}(\Omega)$ a supersolution and $u \geqslant 0$ in $B(y, 4R) \subset \Omega$. Then

$$R^{-n/p}\|u\|_{L^p(B(2R))} \leqslant C\Big(\inf_{y \in B(y,R)} u + k(R)\Big).$$

- *Strong Harnack Inequality:* (R_1), (R_2), $u \in W^{1,2}(\Omega)$ a solution with $u \geqslant 0$. Then

$$\sup_{B(y,R)} u \leqslant C\Big(\inf_{B(y,R)} u + k(R)\Big).$$

- *Strong Maximum Principle:* (R_1), (R_2), (E_3), Ω connected, $u \in W^{1,2}$ a subsolution $Lu \geqslant 0$. If

$$\sup_B u = \sup_\Omega u,$$

  then $u = \text{const}$.
  Proof: Weak Harnack shows $\{u = \sup_\Omega u\}$ is open. $\{u = \sup_\Omega u\}$ is relatively closed in Ω. Therefore $\{u = \sup_\Omega u\} = \Omega$.
  Why is $L\,\text{const} = 0$?
  How dow we know the "relatively closed" part?

- *DeGiorgi/Nash*: (R_1), (R_2), $u \in W^{1,2}$ solution of $Lu = g + \operatorname{div} f$. Then f is locally Hölder and
$$\operatorname{osc}_{B(y,R)} u \leqslant C R^{\alpha}\left(R_0^{-\alpha} \sup_{B(y,R_0)} |u| + k\right)$$
if $0 < R \leqslant R_0$.
Proof:
  -

12 Calculus of Variations

Ω open, bounded.

- Idea: solution u, smooth variation φ, functional I. $\partial_{\varepsilon} I(\, u + \varepsilon\varphi)|_{\varepsilon=0} = 0$. Integrate by parts, φ was arbitrary $\to$ PDE.
- $u: \Omega \to \mathbb{R}^m$ deformation, $Du: \Omega \to \mathbb{R}^{m\times n}$, $F: \mathbb{R}^{m\times n}[\times \mathbb{R}^n] \to \mathbb{R}$.
$$I[u] = \int_{\Omega} F(Du(x), u)\mathrm{d}x.$$
Looking for $\inf_{u\in\mathcal{A}} I[u]$, where $\mathcal{A} = W_0^{1,2}(\Omega)$.
- *Example: Dirichlet's Principle:* Ω open, bounded
$$I[u] = \int_{\Omega}\left(\frac{1}{2}|Du|^2 - gu\right)\mathrm{d}x.$$
  - Bounded below: $\varepsilon a^2 + b^2/\varepsilon$, Sobolev ($2^* > 2$), Hölder as $\|u\|_{L^2} \leqslant \|u\|_{L^{2^*}} |\Omega|^{1/n}$, gives
$$I[u] \geqslant c\|u\|_{W_0^{1,2}}^2 - \frac{1}{2\varepsilon}\|g\|_{L^2}^2.$$
  - Bounded above by $\|u\|_{W_0^{1,2}}^2 + \|g\|_{L^2}$.
  - I wlsc because F convex.
  - strictly convex (unproven) $\Rightarrow$ uniqueness.
- *Weak lower semicontinuity*: $u_k \rightharpoonup u \Rightarrow I[u] \leqslant \liminf_{k\to\infty} I[u_k]$.
- F convex $\Rightarrow I$ wlsc in $W_0^{1,p}(\Omega)$.
Proof: Use representation of convex F as limit of increasing sequence $\{F_N\}$ of piecewise affine functions. Implies $\int F_N(Du_k) \overset{k}{\to} \int F_N(Du)$ (weak convergence ♡ linear/affine functions). Then
$$\int F_N(Du) \overset{F_N \text{ incr.}}{\leqslant} \liminf_{k\to\infty} \int F(Du_k) = \liminf_{k\to\infty} I[u_k]$$
and MCT.
- *Jensen*: $F(\text{w-}*\lim g_k) \leqslant \text{w-}*\lim F(g_k)$.
- *Euler-Lagrange Equation*: Weak form obtained from $i(\tau) = I[u + \tau v]$, where $u = \operatorname{argmin} I[u]$ and looking at $i'(0) = 0$.
$$-\operatorname{div}(F_p(Du)) + F_u(Du, u) = 0.$$
Also $i''(0) \geqslant 0$.
- *Motivation for Convexity*: $\rho(s) =$ 0-1 sawtooth. $\rho' = 1$ a.e.. $v_{\varepsilon}(x) = \varepsilon\zeta(x)\rho(x\cdot\xi/\varepsilon)$.
$$\frac{\partial v_{\varepsilon}}{\partial x_i}(x) \approx \zeta(x)\rho'(x\cdot\xi/\varepsilon) \approx \zeta(x)\xi.$$
Consider $i''(0) \geqslant 0 \Rightarrow \xi^T D^2 F \xi \geqslant 0$ pops out.
- $m = 1 \Rightarrow$ (wlsc $\Leftrightarrow$ convexity).

Proof: "$\Leftarrow$": shown above. "$\Rightarrow$": 2^{nk} cube grid on $[0,1]^n$, $v \in C_c^\infty$.

$$\begin{aligned} u_k(x) &= \frac{1}{2^k} v(2^k(x - \text{cell center})) + z \cdot x. \\ Du_k(x) &= Dv(2^k(x - \text{cell center})) + z. \end{aligned}$$

$u_k \to z \cdot x$, $Du_k \rightharpoonup Du$. Then

$$F(z) \overset{\text{wlsc}}{\leqslant} \liminf_{k\to\infty} \sum_l \int_{Q_l} F(Du_k) = \int_{[0,1]^n} F(z + Dv)$$

Thus $I[u]$ has a minimum at the straight line, and for $i(\tau) = I[u + \tau v]$, $i'(0) = 0$, $i''(0) \geqslant 0$, convexity follows as above.

12.1 Quasiconvexity

$m \geqslant 2$, $\mathcal{A} = W^{1,p} \cap \{u = g\}_{\partial\Omega}$. $1 < p < \infty$. $F \in C^\infty$, $F(A) \geqslant c_1 |A|^p - c_2$.

$$I[u] = \int_\Omega F(Du(x)) \mathrm{d}x$$

- Sawtooth calculation yields *rank-one convexity*

$$(\eta \otimes \xi)^T D^2 F(P)(\eta \otimes \xi)$$

  $\Leftrightarrow F(P + t(\eta \otimes \xi))$ convex in t.

- *Quasiconvexity*: F *quasiconvex*: $\Leftrightarrow \forall A \in \mathbb{R}^{m\times n}$, $v \in C_c^\infty([0,1]^n, \mathbb{R}^m)$:

$$F(A) \leqslant \int_{[0,1]^n} F(A + Dv)$$

- If $|F(A)| \leqslant C(1 + |A|^p)$, then F QC $\Leftrightarrow I$ wlsc.
  - "$\Rightarrow$": Subdivide domain into cubes,

$$\int_\Omega F(Du) \approx \int_\Omega F(\text{affine approx to } Du) \overset{\text{QC}}{\leqslant} \int_\Omega F(Du_k) + \text{errors}.$$

    Use measure theory to keep concentrations (Dirac bumps?) of Du or Du_k away from cube boundaries. Mop up the error terms.
  - "$\Leftarrow$": cubes calculation above.
- *Polyconvex*: F is a convex function of minors of A.
- Convex $\Rightarrow$ PC $\Rightarrow$ QC $\Rightarrow$ R1C (converse false).
  Proof of PC $\Rightarrow$ QC: PC $\Rightarrow$ wlsc (use convex $\Rightarrow$ wlsc argument for each minor). wlsc $\Rightarrow$ QC.
- $|DF(A)| \leqslant C(1 + |A|^{p-1})$.
  Proof: Exploit growth estimate above, and QC $\Rightarrow$ R1C. Use $f(t) = F(A + t(\eta \otimes \xi))$, which is convex $\Rightarrow$ locally Lip $\Rightarrow$ locally $|f'(0)| \leqslant \max |f|$.

12.2 Null Lagrangians, Determinants

- $F(Du)$ is a *null Lagrangian* if E-L

$$\operatorname{div}(DF(Du)) = \partial x_j \big(\partial_{A_{i,j}} F(Du)\big) = 0$$

  holds for every $u \in C^2$.
- F null Langrangian. Then

$$u = \tilde{u} \text{ on } \partial\Omega \quad \Rightarrow \quad I[u] = I[\tilde{u}].$$

  Proof: $i(\tau) := I[\tau u + (1 - \tau) u]$. $i'(\tau) = 0$ by E-L.

- *Cofactor matrix:*
  - $\mathrm{cof}(A_{i,j}) = \det(A_{\setminus i, \setminus j})$.
  - $A^{-1} = \frac{1}{\det A} (\mathrm{cof}\ A)^T$.
  - $\Rightarrow A^T \mathrm{cof}\ A = \det A \cdot \mathrm{Id}$
  - $\Rightarrow \partial_{A_{i,j}} \det(A) = (\mathrm{cof}\ A)_{i,j}$
- $\det(Du)$ is a null Lagrangian, i.e.
  - $\mathrm{div}(D \det(Du)) = \mathrm{div}(\mathrm{cof}(Du)) = 0$
  - Plug and chug if $\det(Du) \neq 0$, otherwise add $\varepsilon \mathrm{Id}$.
- $u_k \rightharpoonup u$ in $W^{1,p}$, $n < p < \infty$. $\Rightarrow \det(Du_k) \rightharpoonup \det(Du)$ in $L^{p/n}$. (Morrey/Reshetnyak)
  - Reduce dimension of problem by one by reducing to "does the cofactor matrix converge"?
  - Use $\det(Du) = \mathrm{div}\Big(\frac{1}{n}\mathrm{cof}(Du)^T u\Big)$.
  - Morrey ($n < p$!) implies uniform boundedness in $C^{0,1-n/p}$, then use A-A to extract uniformly converging subsequence, settling the deal for the leftover u besides the cofactor matrix.
  - (also holds for $p = n$ if $\det(Du_k) \geqslant 0$–no proof.)
- *No Retract Theorem*: $B = B(0,1)$. There is no continuous map $u: \bar{B} \to \partial \bar{B}$ with $u(x) = x$ on ∂B.
  Proof: Suppose there is a retract w. By comparison with Id and identity on the boundary,
  $$\int \det(Dw) = |B|.$$
  OTOH, $|w|^2 = 1 \Rightarrow (Dw)^T w = 0 \Rightarrow \det(Dw) = 0$. Lose smoothness requirement by continuously extending by Id, mollifying and using $B(0,2)$ then.
- *Brouwer's Fixed Point Theorem*: $u: \bar{B} \to \bar{B}$ continuous. $\exists x \in \bar{B}: u(x) = x$.
  Proof: Assume no fixed point. $w: B \to \partial B$ is the point on ∂B hit by the ray from $u(x)$ to x.
  w is a retract because w hits ∂B in x if $x \in \partial B$. w is continuous.
- *Degree of a map:* $u \in W^{1,1}$
  $$\deg(u) = \fint_B \det(Du).$$
  Definable for continuous functions by approximation. Is an integer.

13 Navier-Stokes Equations

G open, $\hat{G} := G \times (0, \infty)$ space-time.

- *Navier-Stokes Equations*:
  $$Du/Dt = (\nu \Delta u - \nabla p) + f,$$
  $(*)$ is the material derivative $D*/Dt = \partial_t * + u \cdot \nabla *$.
- $\nu = 0 \Rightarrow$ Euler equation. $\nu \neq 0 \Rightarrow$ may as well assume $\nu = 1$.
- *Conservation of mass*: $\partial_t \rho + \mathrm{div}(\rho u) = 0$. Assume $D\rho/Dt = 0 \Rightarrow \nabla \cdot u = 0$.
- *Pressures in a smooth incompressible flow are superharmonic*: Take div of NSE.
- *Steady flows*: $u \cdot \nabla u + \nabla p = \nu \Delta u$.
- *Bernoulli's Theorem*: ideal ($\nu = 0$), steady flow $u \cdot \nabla u + \nabla p = 0 \Rightarrow \nabla(u^2/2 + p) = 0$

$\Rightarrow u^2/2 + p = \text{const}$ (still need conservation of mass $\nabla \cdot u = 0$)

- *Vorticity*: $\omega = \operatorname{curl} u$

$$\begin{aligned} \partial_t \omega + \nabla \times (u \cdot \nabla u) &= \Delta \omega, \\ \nabla \cdot u &= 0, \\ \nabla \times u &= \omega. \end{aligned}$$

  In 2D, $\nabla \times (u \cdot \nabla u)$ becomes $u \cdot \nabla u$.

- *Helmholtz Projection*: $P = L^2$-closure $\{\nabla \varphi : \varphi \in C_c^\infty(\mathbb{R}^n)\}$. $P^\perp$ (note P closed!) is *divergence-free*.
  $L^2 = P \oplus P^\perp$
  Example: Divergence-free field from sem. 1 final: (continuous) boundary-normal field matters, (discontinuous) tangential field does not.

- *Weak formulation*:
  - Take $a \in C_c^\infty(\hat{G}, \mathbb{R}^n)$ div-free, dot NSE with it,
  - i. by parts second term, popping the derivative onto *a* u-product, pull apart, one term is zero,
  - $\int a \cdot \nabla p = -\int (\operatorname{div} a) p = 0$

  gives

$$(W1) \quad -\int_{\hat{G}} \partial_t a \cdot u + \nabla a \cdot (u \otimes u) + \Delta a \cdot u \mathrm{d}x \mathrm{d}t = 0$$

$$(W2) \quad \int_{\hat{G}} \nabla \varphi \cdot u = 0 \quad (\varphi \in C_c^\infty(\mathbb{R}^n))$$

  (where $A \cdot B = \operatorname{tr}(A^T B)$)

- $V := \|\cdot\|_V$-closure$\{a \in C_c^\infty(\hat{G}, \mathbb{R}^n), \nabla \cdot a = 0\}$

$$\|a\|_V := \int_{\hat{G}} |a|^2 + |\nabla a|^2 \mathrm{d}x \mathrm{d}t.$$

- Space for ICs: $P_0 := P \cap L^2$-closure $\{C_c^\infty\}$ to replicate $u = 0$ on ∂G.

- *Existence, Energy Inequality:* $u_0 \in P_0^\perp$. $\exists u \in V$:
  - (W1), (W2)
  - *continuous docking to IC*: $\|u(t, \cdot) - u_0\|_{L^2(G)} \to 0$ as $t \to 0$,
  - *energy equality*:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2} = 2\|\nabla u\|_{L^2}.$$

    Equivalently for $t > 0$,

$$\int_G |u(x,t)|^2 + \int_0^t \int_G |\nabla u(x,s)|^2 \mathrm{d}x \mathrm{d}s \leqslant \frac{1}{2} \int_G |u_0(x)|^2 \mathrm{d}x.$$