Nonlinear Filtering using Particles and Quadrature

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Outline

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The Nonlinear Filtering Problem

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- A hidden *Markov process* $\mathbf{x}_t$ for $t = 0, 1, 2, \ldots$ with initial distribution $p(\mathbf{x}_0)$ and transition probability $p(\mathbf{x}_{t+1}|\mathbf{x}_t)$.

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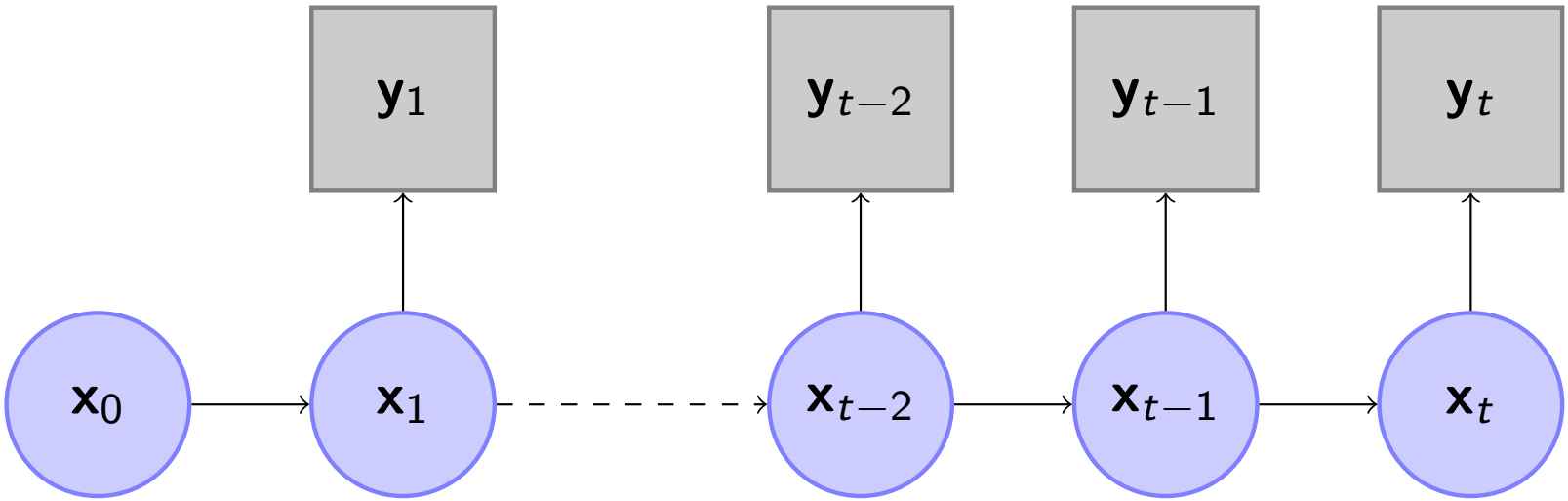
Want:

- (An estimate of) The *posterior distribution* $p(\mathbf{x}_t|\mathbf{y}_{1:t})$.

Process and Observation, schematically



Bayes' Formula

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$p(X|Y)$

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Not usable for on-line processing.

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Not usable for on-line processing. Need a recursive algorithm.

Joint Posterior Update

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Two drawbacks:

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- Don't care about the *joint* posterior–only want $p(\mathbf{x}_t|\mathbf{y}_{1:t})$

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Our Example Process

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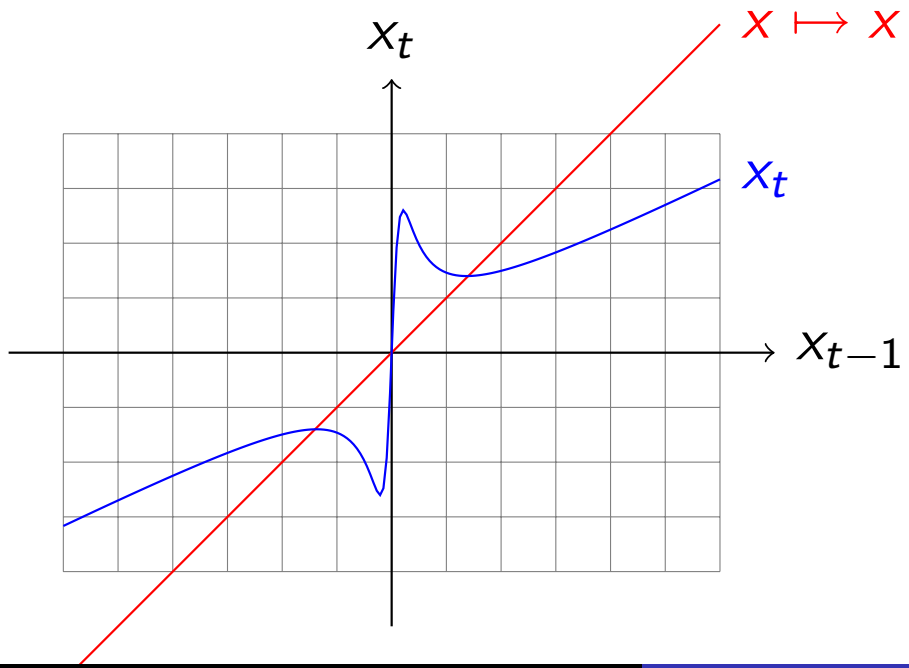

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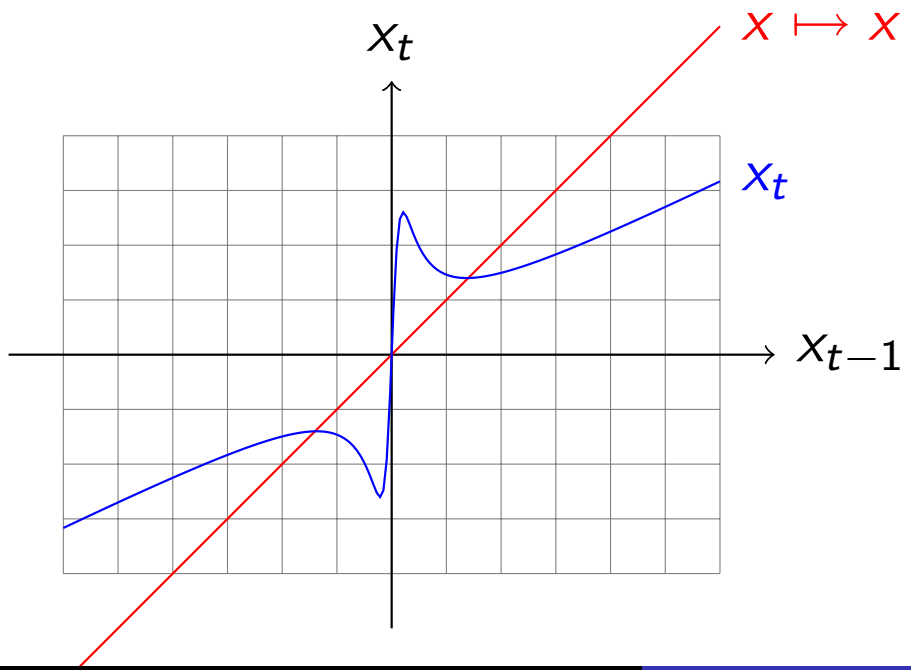
Throughout this presentation, we will stick to the following model:

$$x_t = \frac{1}{2}x_{t-1} + 25\frac{x_{t-1}}{1+x_{t-1}^2} + 8cos(1.2t) + v_t,$$



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where $x_0 \sim \mathcal{N}(0, \sigma_1^2)$, $v_t \sim \mathcal{N}(0, \sigma_v^2)$, $w_t \sim \mathcal{N}(0, \sigma_w^2)$.

Particle Approximations

Assume a discrete approximation of the posterior density

$$\hat{p}(\mathrm{d}\mathbf{x}_t|\mathbf{y}_{1:t}) = \frac{1}{N_t}\sum_{i=1}^{N_t}\delta_{\mathbf{x}_t^{(i)}}(\mathrm{d}\mathbf{x}_t).$$

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Then insert that into the prediction formula:

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$$\hat{p}(\mathrm{d}\mathbf{x}_t|\mathbf{y}_{1:t}) = \frac{1}{N_t}\sum_{i=1}^{N_t}\delta_{\mathbf{x}_t^{(i)}}(\mathrm{d}\mathbf{x}_t).$$

Then insert that into the prediction formula:

$$\hat{p}(\mathrm{d}\mathbf{x}_{t+1}|\mathbf{y}_{1:t}) = \frac{1}{N_t}\sum_{i=1}^{N_t}p(\mathrm{d}\mathbf{x}_{t+1}|\mathbf{x}_t^{(i)})$$

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Problem: (Generally) Nontrivial to sample from!

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Trick: Give each particle a *weight* in addition to its position.

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Any unkown proportionality constant in p or w goes away in the normalization step.

Why and How Importance Sampling Works

Now

$$p(x) \approx \hat{p}(x) := \sum_{i=1}^{N} \tilde{w}^{(i)} \delta(x - x^{(i)}),$$

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$$\int f(x) \frac{p(x)}{\pi(x)} \pi(x) \mathrm{d}x = \int f(x) p(x) \mathrm{d}x.$$

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Not usually easy to sample from$\rightarrow$back to original problem.

Choice of Importance Density

Popular choice:

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→Actually implementable algorithm.

SIS Implementation

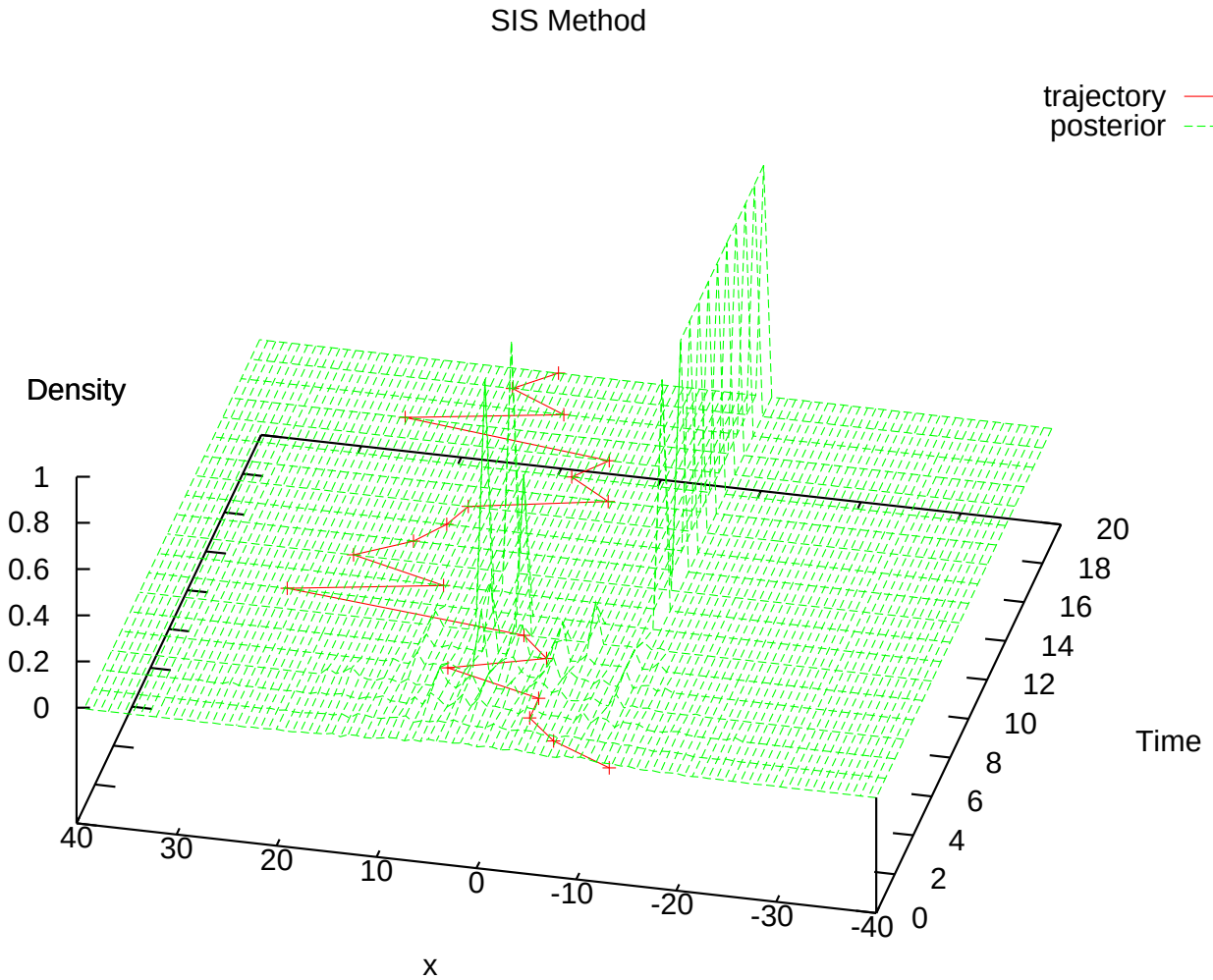

```
def sis(model, n=1000, max_time=20):
    particles = [model.sample_initial() for i in range(n)]
    particle_history = [particles]
    weights = [1/n for i in range(n)]
    weight_history = [weights]

    x = model.sample_initial()
    x_history = [x]

    for t in range(1, max_time):
        x = model.sample_transition(t, x)
        y = model.sample_observation(t, x)
        x_history.append(x)

        evolved_particles = [model.sample_transition(t, xp) for xp
        weights = [weight*model.observation_density(t, y, xtp)
                for weight, xtp in zip(weights, evolved_particles)
        weight_sum = sum(weights)
        weights = [weight/weight_sum for weight in weights]
        particle_history.append(particles)
        weight_history.append(weights)
```

SIS Results



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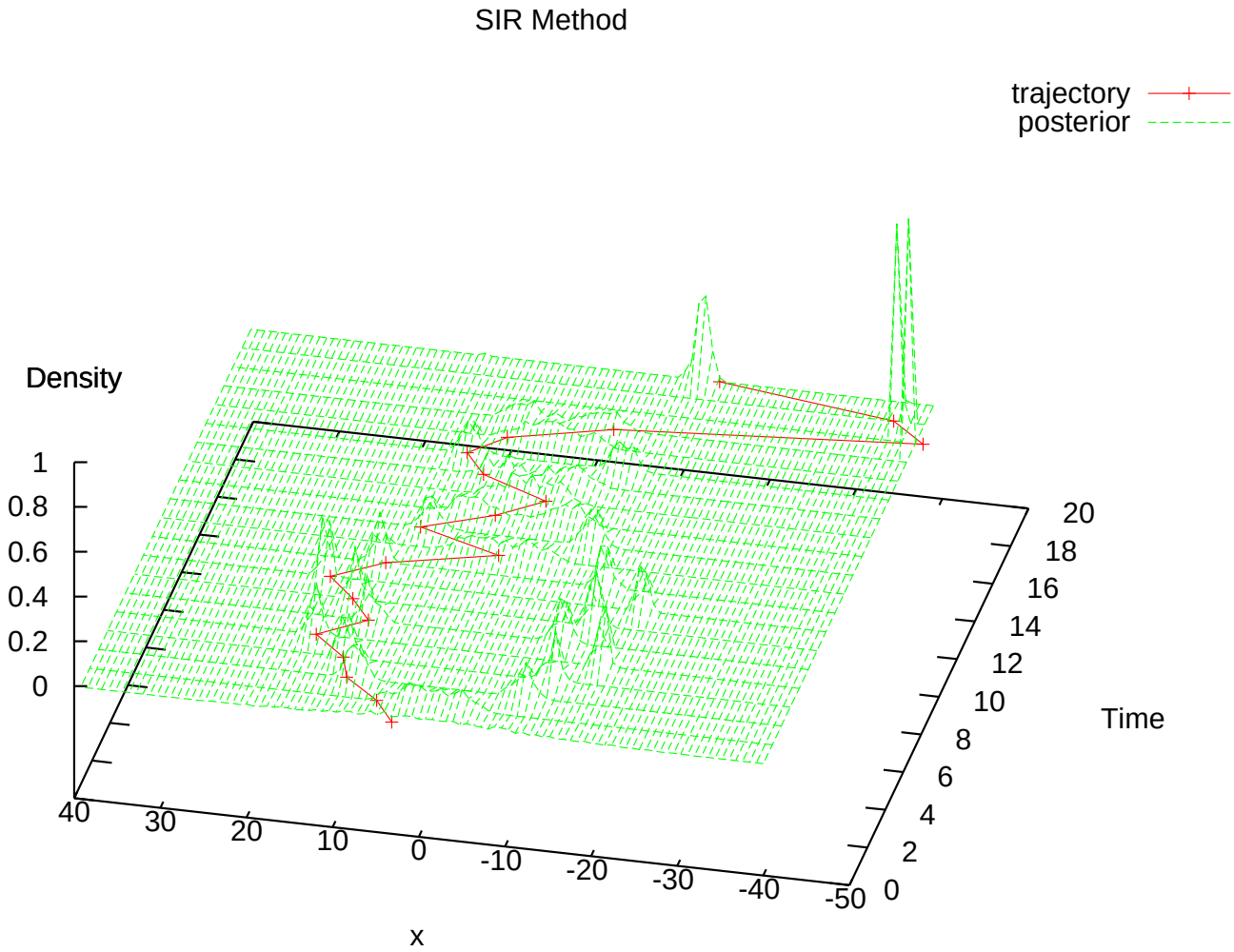
→ *Multinomial Resampling*

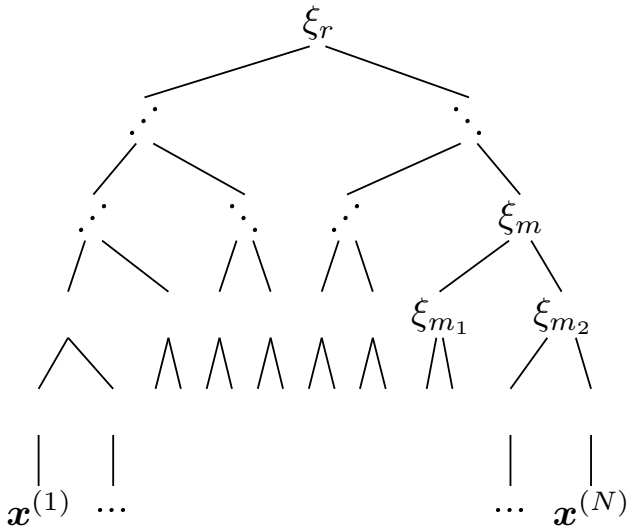
SIR Implementation


```
def sir(model, n=1000, max_time=20):
    particles = [model.sample_initial() for
        i in range(n)]
    particle_history = [particles]

    x = model.sample_initial()
    x_history = [x]

    for t in range(1, max_time):
        x = model.sample_transition(t, x)
        y = model.sample_observation(t, x)
        x_history.append(x)

        evolved_particles = [model.
            sample_transition(t, xp) for xp
            in particles]
        iweights = [model.
            observation_density(t, y, xtp)
            for xtp in evolved_particles]
        particles = sample_discrete(
            evolved_particles, iweights, n)
        particle_history.append(particles)
```

SIR Results



Binary-Tree-based Resampling



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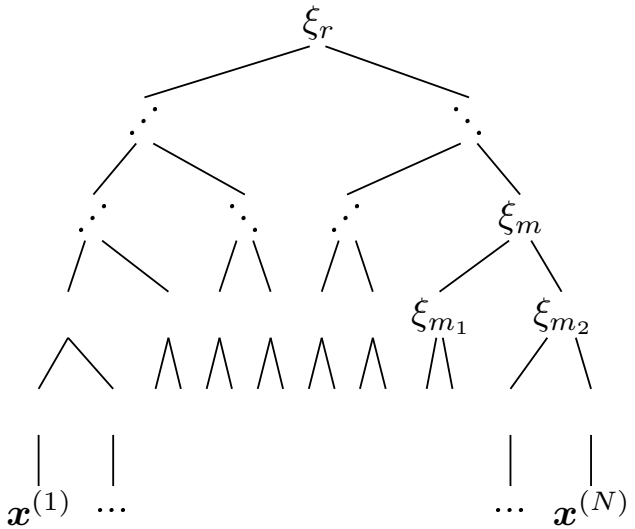

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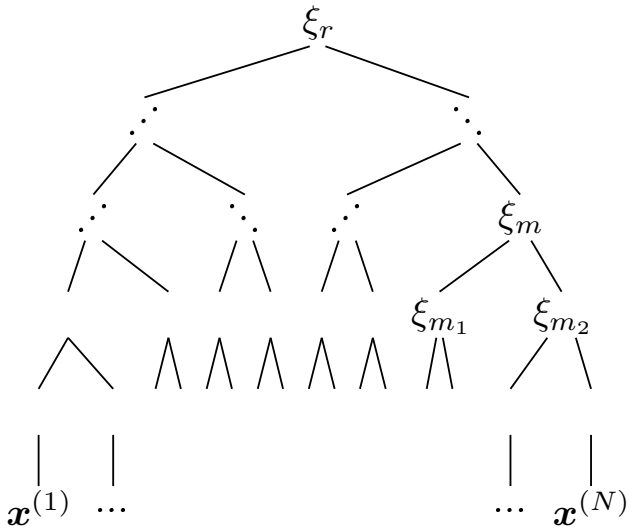

Construct random variables $\xi_m \in \{0, \ldots, N\}$ for each node m, with $\xi_m = \xi_{m_1} + \xi_{m_2}$ such that $E[\xi_m] = w_m$ and

$$\xi_m = \begin{cases} \lfloor w_m \rfloor & \text{with probability } 1 - w_m - \lfloor w_m \rfloor, \\ \lfloor w_m \rfloor + 1 & \text{with probability } \quad w_m - \lfloor w_m \rfloor. \end{cases}$$

A Convergence Result

Theorem (D. Crisan, 2001)

Let the transition kernel $p(\mathbf{x}_t|\mathbf{x}_{t-1})$ *have the* Feller property, *that is for any bounded and continuous* $f : \mathbb{R}^n \to \mathbb{R}$, *the map*

$$\mathbf{x}_{t-1} \mapsto \int f(x) p(\mathrm{d}\mathbf{x}_t|\mathbf{x}_{t-1})$$

must be bounded and continuous. Then as the number of particles $N \to \infty$, *the approximated posterior densities of the SIR method with multinomial resampling*

$$\hat{p}^N(\mathbf{x}_t|\mathbf{y}_{0:t}) \to p(\mathbf{x}_t|\mathbf{y}_{0:t})$$

p-almost surely.

Outline

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Idea: Cleverly reinterpret Gaussian Quadrature as a particle approximation.

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So view the quadrature points $x^{(i)}$ as the particles

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with some quadrature weights $w^{(i)}$ and points $x^{(i)}$.

Consider a probability density function p supported on (A, B).

$$E[f(x)] = \int_A^B f(x)p(x)\mathrm{d}x \approx \sum_{i=1}^N f(x^{(i)})p(x^{(i)})w^{(i)}.$$

So view the quadrature points $x^{(i)}$ as the particles and

$$\tilde{p}(x^{(i)}) := p(x^{(i)})w^{(i)}$$

as the weights.

Construction of the Quadrature Filter

Recall the Update Formula

$$p(\mathbf{x}_t|\mathbf{y}_{1:t}) = \frac{1}{C_t} \cdot p(\mathbf{y}_t|\mathbf{x}_t)p(\mathbf{x}_t|\mathbf{y}_{1:t-1})$$

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Recall the original prediction formula:

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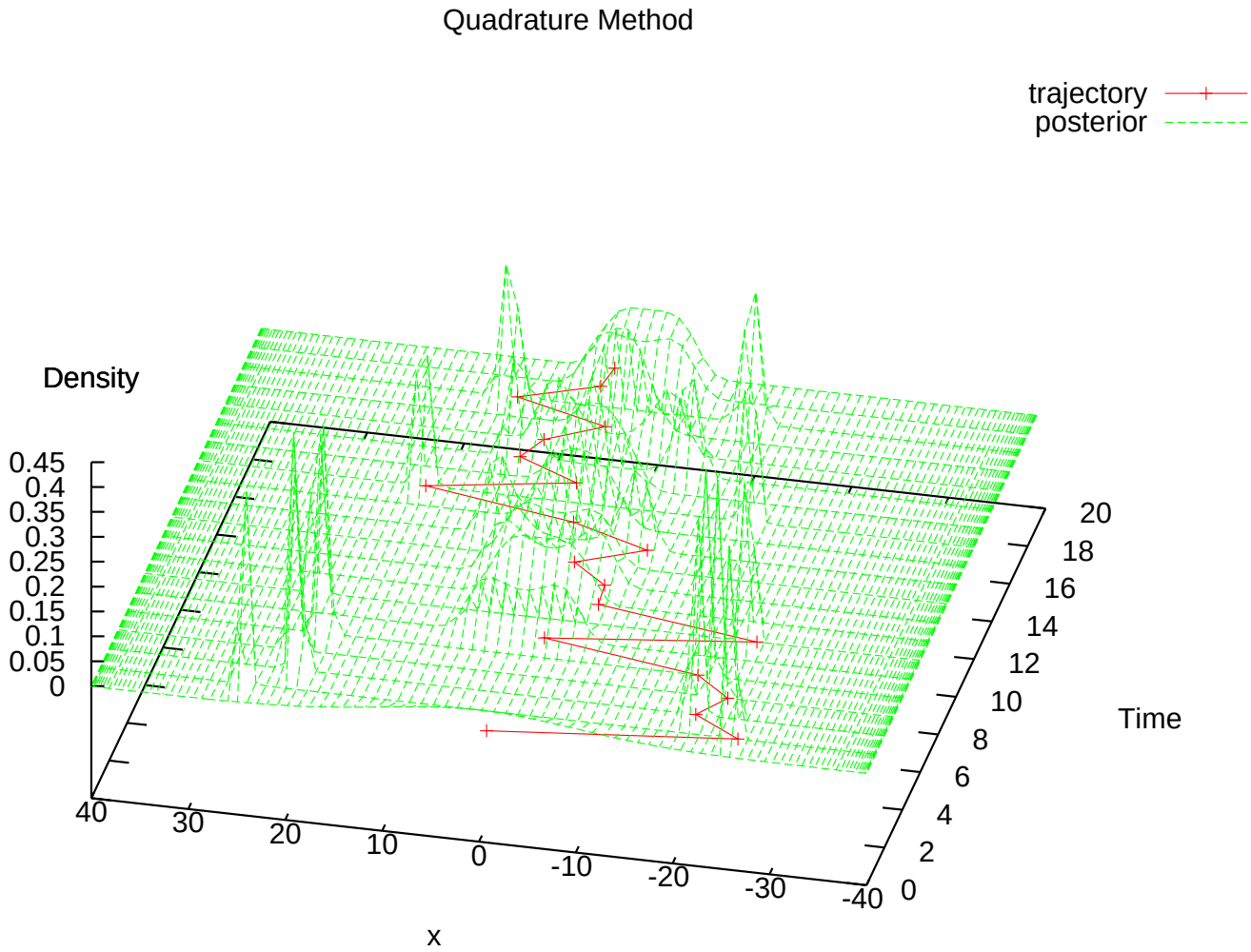
Note: $O(N^2)$ complexity.

Quadrature Filter Results



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- Requires far fewer particles than SMC for comparable accuracy.
- Explicit error estimates available.

Error Estimates: Setup

Error in the log-likelihood of the observation:

$$\delta_t := \log\{\hat{p}(\mathbf{y}_{1:t})\} - \log\{p(\mathbf{y}_{1:t})\} = \sum_{s=1}^{t} \log \hat{C}_s - \sum_{s=1}^{t} \log C_s.$$

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Indicator ignores error in normalization $\hat{C}_t \longrightarrow exp(\delta_t)$ necessary.

Error Propagation Recursion

A nice bit of algebra (see notes) shows

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& \varepsilon(\mathbf{x}_{t+1}|\mathbf{y}_{1:t}) \\
= \quad & C_t^{-1} \sum_i p(\mathbf{x}_{t+1}|\mathbf{x}_t^{(i)}) p(\mathbf{y}_t|\mathbf{x}_t^{(i)}) w_t^{(i)} \underline{\varepsilon(\mathbf{x}_t|\mathbf{y}_{1:t-1})} \\
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Split error into

- Method error (First line)
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  where r_t is the relative error at time t, i.e. the smallest positive real such that

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Proof of Error Bound

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& \varepsilon(\mathbf{x}_{t+1}|\mathbf{y}_{1:t}) \\
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$\longrightarrow$ *Linear* error growth in the *log*-likelihood.

Outline

Continuous-Time Nonlinear Filtering

Have:

- A *Markov diffusion process*

$$\mathrm{d}x^i(t) = b^i(\mathbf{x}(t))\mathrm{d}t + \sigma^{ij}(\mathbf{x}(t))\mathrm{d}W^j(t)$$

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All coefficients are assumed smooth, and $\mathbf{x}_0$ and the Wiener processes $\mathbf{W}$ and $\mathbf{V}$ are assumed to be independent.

The Optimal Continuous-Time Filter

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  - dependency on observation (ξ_α).
- φ_α can be precomputed.

Questions?

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